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NEW YORK:
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# PREFACE.

To many teachers a Key to any mathematical work is not indispensably necessary; but most teachers find that class and school-room duties so fully occupy their time as to make a Key a great convenience, if not a necessity. And teachers of limited acquirements and experience, as well as private learners, can not but find a Key of great service.

In the preparation of this volume two ends have been constantly kept in view: *first*, to give a full and comprehensive solution of every example and problem contained in the New Elementary Algebra, except the most simple and those solved in that work; and, *second*, to give to those who use this Key more extended and enlarged views of the applications of mathematical science, by the introduction of many unique and singular analyses and solutions.

At the bottom of each page of this work will be found the number of the page of the text-book on which the examples and problems may be found. With this exception, the arrangement is so plain as to require no explanation.

# KEY

TO

# ROBINSON'S

# NEW ELEMENTARY ALGEBRA.

---

## ADDITION.

### (48.)

When any number of terms are taken as one term in a vinculum, that is, when the vinculum is the *unit of addition*, let it be represented by V.

30. Let $V = (a + b)$, and the example becomes

$$3aV + 7aV - 5aV + 3aV.$$

The sum is $13aV$ less $5aV$, or $8aV$, which is $8a(a + b)$ the answer.

31. Here $V = (6x + y - z)$,
and we must unite $7V^2$, $-8V^2$, $-2V^2$, and $3V^2$.

The sum is $10V^2$, less $10V^2$, equal to 0, *Ans.*

32. Here $V = (6y + b)$, and the example requires us to unite $11V$ to $-5V$, which makes $6V$. Hence, $6(6y + b)$ or $36y + 6b$, is the sum.

33. Here $V = (z - m)$.

The sum of the plus terms is $18V$, and the sum of the minus terms is $-26V$, and these united, make the result $-8V$. Hence, $-8(2-m)$, *Ans.*

In like manner we solve examples 34 and 35.

(**49**, page 30.)

$$
\begin{array}{lrrr}
5. & 6ab & +\ 12bc & -\ 8cd \\
 & -\ 7ab & -\ 9bc & +\ 3cd \\
 & -\ 2ab & -\ 5bc & +\ 12cd \\
\hline
\textit{Ans.} & -\ 3ab & -\ 2bc & +\ 7cd
\end{array}
$$

$$
\begin{array}{lrrr}
6. & 9b^2 & -\ 3ac & +\ d \\
 & 4b^2 & -\ 4ac & +\ 7d \\
 & -\ 4b^2 & +\ 6ac & +\ 3d \\
 & 5b^2 & -\ 2ac & -\ 12d \\
 & 4b^2 & & -\ d \\
\hline
\textit{Ans.} & 18b^2 & -\ 3ac & -\ 2d
\end{array}
\qquad
\begin{array}{lrrrr}
7. & 7ab & -\ m^2 & +\ q & \\
 & -\ 4ab & -\ 5m^2 & -\ 3q & \\
 & 12ab & +\ 14m^2 & & -\ z \\
 & & -\ 6m^2 & -\ 2q & \\
\hline
\textit{Ans.} & 15ab & +\ 2m^2 & -\ 4q & -\ z
\end{array}
$$

Examples 8, 9, and 10, can be added mentally.

(Page 31.)

$$
\begin{array}{lrrrr}
11. & 15a^2 & -\ 8b^2c & +\ 32a^2c^3 & -\ 12bc \\
 & -\ 4a^2 & +\ 19b^2c & +\ 11a^2c^3 & +\ 2bc \\
 & a^2 & -\ 12b^2c & -\ 29a^2c^3 & +\ 5bc \\
 & & +\ b^2c & +\ 9a^2c^3 & -\ 14bc \\
\hline
 & 12a^2 & & +\ 23a^2c^3 & -\ 19bc \quad \textit{Ans.}
\end{array}
$$

$$
\left.
\begin{array}{lrrrr}
12. & 5a^3b^2 & -\ 8a^2b^3 & +\ x^2y & +\ xy^2 \\
 & -\ 7a^3b^2 & +\ 4a^2b^3 & +\ 6x^2y & -\ 3xy^2 \\
 & 3a^3b^2 & +\ 3a^2b^3 & -\ 3x^2y & +\ 5xy^2 \\
 & -\ a^3b^2 & +\ 2a^2b^3 & -\ 3x^2y & -\ 3xy^2 \\
\hline
\textit{Ans.} & 0 & a^2b^3 & +\ x^2y & 0
\end{array}
\right\}
\text{Here are 4 examples, blended in one.}
$$

13.
$$\begin{array}{rrrrr}
72ax^4 & & -8ay^3 & & \\
-38ax^4 & -3ay^4 & +7ay^3 & & \\
 & +12ay^4 & +5ay^3 & +8 & \\
-34ax^4 & -9ay^4 & -6ay^3 & +12 & \\
\hline
 & \textit{Ans.} & -2ay^3 & +20 &
\end{array}$$

14.
$$\begin{array}{lrrrrr}
 & 7x^2 & -5cx & +14mg & & \\
 & -3x^2 & +4cx & -17mg & -pq & \\
 & 4x^2 & & +12mg & +3pq & -z \\
 & & 2cx & -7mg & -2pq & \\
 & 3x^2 & -2cx & -mg & -4pq & +3z \\
\hline
\textit{Sum,} & 11x^2 & -cx & +mg & -4pq & +2z
\end{array}$$

15.
$$\begin{array}{lrrrr}
 & 7m & +3n & -11p & \\
 & -11m & -9n & & +3a \\
 & -4m & +8n & +5p & \\
 & -m & +6n & +3p & \\
\hline
\textit{Ans.} & -9m & +8n & -3p & +3a
\end{array}$$

Examples 16, 17, and 18 can be added mentally.

19.
$$\begin{array}{lrr}
 & 6V & +2c \\
 & -5V & +7c \\
 & 3V & -4c \\
 & 4V & +c \\
\hline
\textit{Sum,} & 8(m^2-n) & +6c
\end{array}$$

20.
$$\begin{array}{lrr}
 & 2aV & -3mz^2 \\
 & 4aV & -5mz^2 \\
 & 5aV & +7mz^2 \\
\hline
\textit{Ans.} & 11a(x-y^2) & -mz^2
\end{array}$$

21.
$$\begin{array}{lrrrr}
 & 8ax & +2V & +3b & \\
 & 9ax & +6V & -9b & \\
 & -7ax & -8V & +6b & +11x \\
\hline
\textit{Ans.} & 10ax & +0 & +0 & +11x
\end{array}$$

## SUBTRACTION.

(**53**, page 38.)

20. $\begin{array}{r}13a^2b^3 + 11a - 5a^2 + 6b \\ -10a^2b^3 + \phantom{1}7a - 5a^2 + 6b \\ \hline \end{array}$

*Diff.* $23a^2b^3 + 4a$

23. $\begin{array}{l}2ab + b^2 - 4c + bc - b \\ \phantom{2ab + {}} b^2 - c \phantom{{} + bc - b} + 3a^2 \\ \hline \end{array}$

*Diff.* $2ab - 3c + bc - b - 3a^2$

24. $\begin{array}{l}a^3 + 3b^2c + ab^2 - abc \\ \phantom{a^3 + 3b^2c + {}} ab^2 - abc + b^3 \\ \hline \end{array}$

*Diff.* $a^3 + 3b^2c - b^3$

25. $\begin{array}{l}5x^2y - 3bx + c \\ 3x^2y + 2bx \phantom{{}+c} + c^2 \\ \hline \end{array}$

*Diff.* $2x^2y - 5bx + c - c^2$

26. $\begin{array}{l}4m^2 - m + 2cx - y^2 \\ -3m^2 - m + cx + y^2 \\ \hline \end{array}$

*Diff.* $7m^2 + cx - 2y^2$

27. $\begin{array}{l}3a^2 + 0 \\ 0 + 3a - x + b \\ \hline \end{array}$

*Diff.* $3a^2 - 3a + x - b$

**Examples 28, 29, 30, and 31, require but mental operations.**

(**54.**)

9. From $3a^2 - by$
   Take $2a^2 - cy$
   *Diff.* $a^2 + (c - b)y$

(**31 - 39**)

10. $$\begin{array}{l} 5acx^4 + 20ax^3y^2 - 25m \\ 3acx^4 + 12ax^3y^2 - 20m \\ \hline \end{array}$$

*Ans.* $2acx^4 + \ \ 8ax^3y^2 - \ \ 5m$;

Or, $2ax^3(cx + 4y^2) \ - \ 5m$

11. $$\begin{array}{lcl} (2a + b + c)x & = & 2ax + bx + cx \\ (a + b)x & = & ax + bx \\ \hline \end{array}$$

*Diff.* $(a + c)x \quad = \quad ax + cx$

The remaining examples in this article are readily solved mentally.

---

## MULTIPLICATION.

( **59**, page 46. )

6. $$\begin{array}{rl} & 3a^2 - \ \ 2ab - b^2 \\ & 2a \ - \ \ 4b \\ \hline & 6a^3 - \ \ 4a^2b - 2ab^2 \\ & \quad - 12a^2b + 8ab^2 + 4b^3 \\ \hline \textit{Whole prod.} & 6a^3 - 16a^2b + 6ab^2 + 4b^3 \end{array}$$

7. $$\begin{array}{l} x^2 - xy + y^2 \\ x + y \\ \hline x^3 - x^2y + xy^2 \\ \quad + x^2y - xy^2 + y^3 \\ \hline \end{array}$$

*Ans.* $x^3 \qquad\qquad\qquad + y^3$; that is $x^3 + y^3$

$$\begin{array}{rl}
8. & 3a + 4c \\
 & 2a - 5c \\ \hline
 & 6a^2 + 8ac \\
 & \qquad - 15ac - 20c^2 \\ \hline
\textit{Prod.} & 5a^2 - 7ac - 20c^2
\end{array}$$

$$\begin{array}{rl}
9. & a^2 + ay - y^2 \\
 & a - y \\ \hline
 & a^3 + a^2y - ay^2 \\
 & \qquad - a^2y - ay^2 + y^3 \\ \hline
\textit{Ans.} & a^3 - 2ay^2 + y^3
\end{array}$$

$$\begin{array}{rl}
10. & a^2 + ay + y \\
 & a - y \\ \hline
 & a^3 + a^2y + ay^2 \\
 & \qquad - a^2y - ay^2 - y^3 \\ \hline
\textit{Prod.} & a^3 - y^3
\end{array}$$

$$\begin{array}{rl}
11. & a^2 - ay + y^2 \\
 & a + y \\ \hline
 & a^3 - a^2y + ay^2 \\
 & \qquad + a^2y - ay^2 + y^3 \\ \hline
\textit{Prod.} & a^3 + y^3
\end{array}$$

$$\begin{array}{rl}
12. & a^3 + a^2y + ay^2 + y^3 \\
 & a - y \\ \hline
 & a^4 + a^3y + a^2y^2 + ay^3 \\
 & \qquad - a^3y - a^2y^2 - ay^3 - y^4 \\ \hline
\textit{Ans.} & a^4 \qquad\qquad\qquad\qquad - y^4
\end{array}$$

$$\begin{array}{rl}
13. & y^2 - y + 1 \\
 & y + 1 \\ \hline
 & y^3 - y^2 + y \\
 & \qquad + y^2 - y + 1 \\ \hline
\textit{Prod.} & y^3 + 1
\end{array}$$

$$\begin{array}{rl}
14. & x^2 + y^2 \\
 & x^2 - y^2 \\
 & x^4 + x^2y^2 \\
 & \qquad - x^2y^2 - y^4 \\ \hline
\textit{Prod.} & x^4 - y^4
\end{array}$$

$$\begin{array}{rl}
15. & a^2 - 3a + 8 \\
 & a + 3 \\ \hline
 & a^3 - 3a^2 + 8a \\
 & \qquad + 3a^2 - 9a + 24 \\ \hline
\textit{Prod.} & a^3 - a + 24
\end{array}$$

16. $b^4 + b^2x^2 + x^4$

$b^2 - x^2$

$b^6 + b^4x^2 + b^2x^4$

$\quad - b^4x^2 - b^2x^4 - x^6$

*Prod.* $b^6 - x^6$

17. $a^2 + 2b$

$2a^2 - 4b$

$2a^4 + 4a^2b$

$\quad - 4a^2b - 8b^2$

*Ans.* $2a^4 - 8b^2$

18. $x^6 + x^4 + x^2$

$x^2 - 1$

$x^8 + x^6 + x^4$

$\quad - x^6 - x^4 - x^2$

*Ans.* $x^8 - x^2$

19. $m + n$

$9m - 9n$

$9m^2 - 9mn$

$\quad + 9mn - 9n^2$

*Prod.* $9m^2 - 9n^2$

20. $2x^2 + xy - 2y^2$

$3x - 3y$

$6x^3 + 3x^2y - 6xy^2$

$\quad - 6x^2y - 3xy^2 + 6y^3$

*Ans.* $6x^3 - 3x^2y - 9xy^2 + 6y^3$

21. $m^2 - 3m - 7$

$m - 2$

$m^3 - 3m^2 - 7m$

$\quad - 2m^2 + 6m + 14$

*Ans.* $m^3 - 5m^2 - m + 14$

22.
$$
\begin{array}{l}
a^4 - 2a^3c + 4a^2c^2 - 8ac^3 + 16c^4 \\
a + 2c \\
\hline
a^5 - 2a^4c + 4a^3c^2 - 8a^2c^3 + 16ac^4 \\
\quad + 2a^4c - 4a^3c^2 + 8a^2c^3 - 16ac^4 + 32c^5 \\
\hline
\end{array}
$$

*Ans.* $a^5 + 32c^5$

23.
$$
\begin{array}{l}
x^3 - 3x^2 + 3x - 9 \\
x + 3 \\
\hline
x^4 - 3x^3 + 3x^2 - 9x \\
\quad + 3x^3 - 9x^2 + 9x - 27 \\
\hline
\end{array}
$$

*Ans.* $x^4 - 6x^2 - 27$

24.
$$
\begin{array}{l}
m^4 - m^3 + m^2 - m + 1 \\
m + 1 \\
\hline
m^5 - m^4 + m^3 - m^2 + m \\
\quad + m^4 - m^3 + m^2 - m + 1 \\
\hline
\end{array}
$$

*Ans.* $m^5 + 1$

25. Same as 24, except the product by $m$ will be all plus, and the product by $-1$, all minus, and the result is $m^5 - 1$.

26.
$$
\begin{array}{l}
2a^3 + 5ac^2 - 2c^3 \\
2a^3 - 5ac^2 + 2c^3 \\
\hline
4a^6 + 10a^4c^2 - 4a^3c^3 \\
\qquad - 10a^4c^2 - 25a^2c^4 + 10ac^5 \\
\qquad\qquad + 4a^3c^3 + 10ac^5 - 4c^6 \\
\hline
\end{array}
$$

*Ans.* $4a^6 - 25a^2c^4 + 20ac^5 - 4c^6$

27. $$\begin{array}{l} m^2 + 2c \\ m^2 - 5c \\ \hline m^4 + 2cm^2 \\ \qquad - 5cm^2 - 10c^2 \\ \hline m^4 - 3cm^2 - 10c^2 \end{array}$$
*Ans.* $m^4 - 3cm^2 - 10c^2$

**All other examples, as far as Ex. 10 on page 52, are so simple that we omit them.**

( **62**, page 52. )

**10. We observe that the 1st and 3d factors are equal; therefore, we square one of them mentally, and obtain**

$$\begin{array}{l} 9x^2 - 6mx + m^2 \\ x^2 + m^2 \\ \hline 9x^4 - 6mx^3 + \quad m^2x^2 \\ \qquad\qquad + \; 9m^2x^2 - 6m^3x + m^4 \\ \hline 9x^4 - 6mx^3 + 10m^2x^2 - 6m^3x + m^4 \end{array}$$

11. $$\begin{array}{l} 18a + 27x \\ 2a + 3x \\ \hline 36a^2 + 54ax \\ \qquad\quad 54ax + 81x^2 \\ \hline 36a^2 + 108ax + 81x^2 \end{array}$$
*Ans.* $36a^2 + 108ax + 81x^2$

Or, $$\begin{array}{l} 4a^2 + 12ax + 9x^2 \\ 9 \\ \hline 36a^2 + 108ax + 81x^2 \end{array}$$

12. **Mentally, we find the result to be $49c^2d^4 - 16y^2z^2$.**

**Examples 13 and 14 are similar, and may be performed mentally.**

15. We commence with the last two factors, and mentally we discover their product to be $m^2-1$. Now the product of this, and the next preceding factor, we in like manner discover to be $m^4-1$. This into the next, gives us $m^8-1$; this with the next, $m^{16}-1$, and this into the last remaining factor, will give $m^{32}-1$, *Ans.*

16. $(m-1)+(z-1)$

$\underline{(m+1)-(z+1)}$

$m^2-1+(m+1)(z-1)-(m-1)(z+1)-z^2+1$

This is $m^2-z^2+mz-m+z-1-mz-m+z+1$,

Which is $m^2-z^2-2m+2z$, *Ans.*

---

## DIVISION.

( **67**, page 58. )

When all the terms in any example have a common vinculum, represent it by V.

21. Let $V=a+c$.

Then we are required to divide $2aV+V^2$ by V. The quotient must be $2a+V$, which is $2a+a+c$, or $3a+c$, *Ans.*

22. $5cV-V^2$. Divide by V. *Quot.* $5c-V$.

$5c$

$\underline{3m-2c}$

*Diff.* $7c-3m$, *Ans*

23. Can now be done mentally.

(68.)

2. We discover that the dividend is the square of the divisor. Hence the quotient must be $V^2$ divided by $V$, which is $V$, or $a + x$.

3. The dividend is the cube of $a - y$. Hence the quotient is $(a-y)^2 = a^2 - 2ay + y^2$.

4. The dividend is $a^3 + 3a^2b + 3ab^2 + b^3 + 2a^2b + 2ab^2$, which is $(a + b)^3 + 2ab(a + b)$, and this, divided by $(a + b)$, will give the quotient $(a+b)^2+2ab$; or $a^2+4ab+b^2$, *Ans.*

5.
$$
\begin{array}{l|l}
x^3 - 3x^2z + z^3 & x - z \\
\cline{2-2}
\underline{x^3 - \phantom{3}x^2z} & x^2 - 2xz - 2z^2 - \dfrac{z^3}{x-z},\ \textit{Ans.} \\
\quad -2x^{2}2 & \\
\quad \underline{-2x^{2}2 + 2xz^2} & \\
\qquad\qquad -2xz^2 + \phantom{2}z^3 & \\
\qquad\qquad \underline{-2xz^2 + 2z^3} & \\
\qquad\qquad\qquad\quad -\ z^3 &
\end{array}
$$

6.
$$
\begin{array}{l|l}
a^3 + 2a^2b + 2ab^2 + b^3 & a^2 + ab + b^2 \\
\cline{2-2}
\underline{a^3 + \phantom{2}a^2b + \phantom{2}ab^2} & a + b,\ \textit{Ans.} \\
\qquad\quad a^2b + \phantom{2}ab^2 + b^3 & \\
\qquad\quad \underline{a^2b + \phantom{2}ab^2 + b^3} &
\end{array}
$$

7. The dividend is the cube of $x - 3$, and this, divided by $x - 3$, will give the square of $x - 3$; or $x^2 - 6x + 9$ for the quotient.

(69)

8. In this example we are to divide $6x^4 - 96$ by $6x - 12$. Dividing each factor by the common factor 6, and we have $x^4 - 16$ to divide by $x - 2$.

But $x^4 - 16 = (x^2+4)(x^2-4)$. Also $x^2 - 4 = (x+2)(x-2)$.
Hence we must divide $(x^2+4)(x+2)(x-2)$ by $(x-2)$.
The quotient is $(x^2+4)(x+2) = x^3 + 2x^2 + 4x + 8$, *Ans.*

*By the Common Method.*

$$
\begin{array}{l}
6x - 12)\,6x^4 - 96\,(x^3 + 2x^2, \text{ \&c.} \\
\qquad\qquad\ \underline{6x^4 - 12x^3} \\
\qquad\qquad\qquad\quad 12x^3 - 96
\end{array}
$$

9. Dividing by the common factor $3a$, and we have

$$
\begin{array}{l}
a - 1)\,2a^3 + 3a - 5\,(2a^2 + 2a + 5, \textit{ Ans.} \\
\qquad\ \underline{2a^3 - 2a^2} \\
\qquad\qquad\ 2a^2 + 3a \\
\qquad\qquad\ \underline{2a^2 - 2a} \\
\qquad\qquad\qquad\ 5a - 5 \\
\qquad\qquad\qquad\ \underline{5a - 5}
\end{array}
$$

10. Omitting the common factor $x$ in each term, we have

$$
\begin{array}{l}
5x - 4)\,25x^4 - x^2 - 2x - 8\,(5x^3 + 4x^2 + 3x + 2, \textit{ Ans.} \\
\qquad\quad \underline{25x^4 - 20x^3} \\
\qquad\qquad\quad 20x^3 - x^2 \\
\qquad\qquad\quad \underline{20x^3 - 16x^2} \\
\qquad\qquad\qquad\quad 15x^2 - 2x \\
\qquad\qquad\qquad\quad \underline{15x^2 - 12x} \\
\qquad\qquad\qquad\qquad\quad 10x - 8 \\
\qquad\qquad\qquad\qquad\quad \underline{10x - 8}
\end{array}
$$

11. Omitting 2, the common factor, and we have

$$
\begin{array}{l}
3a+2b)\,9a^2-4b^2\,(3a-2b,\ \textit{Ans.} \\
\qquad\quad 9a^2+6ab \\
\hline
\qquad\qquad\quad -6ab-4b^2 \\
\qquad\qquad\quad -6ab-4b^2 \\
\hline
\end{array}
$$

12.
$$
\begin{array}{l}
x-8)\,2x^3-19x^2+26x-16\,(2x^2-3x+2,\ \textit{Ans.} \\
\qquad\ 2x^3-16x^2 \\
\hline
\qquad\quad -3x^2+26x \\
\qquad\quad -3x^2+24x \\
\hline
\qquad\qquad\qquad 2x-16 \\
\qquad\qquad\qquad 2x-16 \\
\hline
\end{array}
$$

13.
$$
\begin{array}{l}
y+1)\,y^5+1\,(y^4-y^3+y^2-y+1,\ \textit{Ans.} \\
\qquad\ y^5+y^4 \\
\hline
\qquad\quad -y^4+1 \\
\qquad\quad -y^4-y^3 \\
\hline
\qquad\qquad\quad y^3+1
\end{array}
$$

We perceive that the forms of $y$ must diminish 1 each term, and that every other sign is *minus*.

Hence, the quotient of $y^{20}+1$, divided by $y+1$, must be $y^{19}-y^{18}+y^{17}-y^{16}+y^{15}-y^{14}+y^{13}-y^{12}+y^{11}-y^{10}+y^9-y^8+y^7-y^6+y^5-y^4+y^3-y^2+y-1$.

14. Divide $y^6-1$ by $y-1$. The first term of the quotient must be $y^5$, every term must be plus, and the powers must diminish by 1 each term. Therefore, the quotient must be $y^5+y^4+y^3+y^2+y+1$.

15. Divide $x^2 - a^2$ by $x - a$. The dividend is the difference between two squares, and it is, therefore, the product of $x - a$ into $x + a$. Hence, $(x + a)$ must be the quotient.

16.
$$
\begin{array}{l}
3a^2 - 1\ )\ 6a^3 - 3a^2b - 2a + b\ (\ 2a - b,\ \textit{Ans.} \\
\qquad\qquad\ \underline{6a^3 \qquad\quad - 2a} \\
\qquad\qquad\qquad - 3a^2b + b \\
\qquad\qquad\qquad \underline{- 3a^2b + b}
\end{array}
$$

17. $y^3 - 3y^2x + 3xy^2 - x^3\ )\ y^6 - 3y^4x^2 + 3y^2x^4 - x^6\ (\ y^3 + \&c.$ This divisor is obviously $(y - x)^3$, and the dividend is $(y^2 - x^2)^3$, or $(y - x)^3\ (y + x)^3$. Therefore, the quotient must be $(y + x)^3$, which is $y^3 + 3y^2x + 3yx^2 + x^3$, *Ans.*

18.
$$
\begin{array}{l}
8a^2b^3 + 5ab^4\ )\ 64a^4b^6 - 25a^2b^8\ (\ 8a^2b^3 - 5ab^4,\ \textit{Ans.} \\
\qquad\qquad\qquad \underline{64a^4b^6 + 40a^3b^7} \\
\qquad\qquad\qquad\qquad\quad - 40a^3b^7 - 25a^2b^8 \\
\qquad\qquad\qquad\qquad\quad \underline{- 40a^3b^7 - 25a^2b^8}
\end{array}
$$

19.
$$
\begin{array}{l}
a - x\ )\ 2a^4 - 2x^4\ (\ 2a^3 + 2a^2x \\
\qquad\quad\ \underline{2a^4 - 2a^3x} \\
\qquad\qquad\qquad 2a^3x
\end{array}
$$

Now we have the form of the quotient; 2 will be in every term; the exponent of $a$ will diminish by 1, and that of $x$ increase by 1; therefore, the following terms must be $2ax^2 + 2x^3$.

Thus, we may divide the difference of any two even powers of $a$ and $x$, by $a - x$, and write out the quotient at once.

Thus, the quotient of $5a^{10} - 5x^{10}$ divided by $a - x$, must be $5a^9 + 5a^8x + 5a^7x^2 + 5a^6x^3 + 5a^5x^4 + 5a^4x^5 + 5a^3x^6 + 5a^2x^7 + 5ax^8 + 5x^9$.

The quotient commences with $5a^9$, and ends with $5x^9$; and the sum of the exponents of $a$ and $x$, in each term, make 9. Hence, the quotient of $ba^n - bx^n$ divided by $a - x$, must be $ba^{n-1} + ba^{n-2}x + ba^{n-3}x^2 +$, and so on to $n$ terms.

21. This is, $(a - x)^3$ divided by $(a - x)$. Whence the quotient is $(a - x)^2 = a^2 - 2ax + x^2$.

22. This is the same as Ex. 13, by changing $y$ to $a$.

23. This is similar to Ex. 14.

24.
$$\begin{array}{l|l}
48a^3-92a^2x-40ax^2-100x^3 & 3a-5x \\
\underline{48a^3-80a^2x} & \overline{16a^2-4ax-20x^2}, \textit{ Ans.} \\
\quad -12a^2x-40ax^2 & \\
\quad \underline{-12a^2x+20ax^2} & \\
\qquad\quad -60ax^2+100x^3 & \\
\qquad\quad \underline{-60ax^2+100x^3} &
\end{array}$$

25.
$$\begin{array}{l|l}
4d^4-9d^2+6d-1 & 2d^2+3d-1 \\
\underline{4d^4+6d^3-2d^2} & \overline{2d^2-3d+1}, \textit{ Ans.} \\
\quad -6d^3-7d^2+6d & \\
\quad \underline{-6d^3-9d^2+3d} & \\
\qquad\quad 2d^2+3d-1 &
\end{array}$$

26. $$\begin{array}{l} 6a^4+4a^3x-9a^2x^2-3ax^3+2x^4\ (\ \underline{2a^2+2ax-\ x^2} \\ \underline{6a^4+6a^3x-3a^2x^2} \qquad\qquad\qquad 3a^2-\ ax-2x^2 \\ \quad -2a^3x-6a^2x^2-3ax^3 \\ \quad \underline{-2a^3x-2a^2x^2+\ ax^3} \\ \qquad\qquad -4a^2x^2-4ax^3+2x^4 \\ \qquad\qquad \underline{-4a^2x^2-4ax^3+2x^4} \end{array}$$

27. $$\begin{array}{l} 3a^4-8a^2b^2+3a^2c^2+5b^4-3b^2c^2\ (\ \underline{a^2-\ b^2} \\ \underline{3a^4-3a^2b^2} \qquad\qquad\qquad\qquad 3a^2-5b^2+3c^2 \\ \quad -5a^2b^2+5b^4 \\ \quad \underline{-5a^2b^2+5b^4} \\ \qquad\qquad 3a^2c^2-3b^2c^2 \\ \qquad\qquad \underline{3a^2c^2-3b^2c^2} \end{array}$$

28. $$\begin{array}{l} x+2y\ )\ 2x^2+7xy+6y^2\ (\ 2x+3y,\ \textit{Ans.} \\ \qquad\qquad \underline{2x^2+4xy} \\ \qquad\qquad\qquad 3xy+6y^2 \\ \qquad\qquad\qquad \underline{3xy+6y^2} \end{array}$$

29. $$\begin{array}{l} x+5n\ )\ 2mx+3nx+10mn+15n^2\ (\ 2m+3n \\ \qquad\qquad \underline{2mx\qquad\quad +10mn} \\ \qquad\qquad\qquad 3nx\qquad\quad +15n^2 \\ \qquad\qquad\qquad 3nx\qquad\quad +15n^2 \end{array}$$

30. $$\begin{array}{l} d^4-3d^2c-10c^2\ (\ \underline{d^2-5c} \\ \underline{d^4-5d^2c} \qquad\qquad d^2+2c,\ \textit{Quot.} \\ \qquad 2d^2c-10c^2 \\ \qquad \underline{2d^2c-10c^2} \end{array}$$

31. $m^2 - c^2 + 2cz - z^2$ ( $m + c - z$
$m^2 + cm - mz$ $\quad m - c + z$, *Quot.*
$- cm + mz - c^2 + 2cz$
$- cm \quad - c^2 + cz$
$mz + cz - z^2$
$mz + cz - z^2$

32. $y^5 + 32z^5$ ( $y + 2z$
$y^5 - 2y^4z \quad y^4 - 2y^3z + 4y^2z^2$
$- 2y^4z$
$- 2y^4z - 4y^3z^2$
$4y^3z^2$

Now we observe that the terms in the quotient have a regular *law* of progression. The numeral coefficients double, each term; the exponents of $y$ decrease by 1, and those of $z$, increase by 1, each term. Therefore, the two remaining terms must be $- 8yz^3 + 16z^4$.

33 and 34. Using V as already taught, these examples can be performed mentally.

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## FACTORING.

(**87**, page 71.)

11. Factor $x^4 - y^4$. When the exponents are even, and the two quantities are connected by the minus sign, we can divide the exponents by 2; in short, we know by **61**, that two factors of $x^4 - y^4$ are $(x^2 + y^2)(x^2 - y^2)$. In like manner, the factors of $x^2 - y^2$ are $(x + y)(x - y)$, and writing these

last two factors in place of their product, we have $(x^2 + y^2)$ $(x + y)$ $(x - y)$ for the required factors.

12. Factor $x^8 - z^8$. 1st result $(x^4 + z^4)$ $(x^4 - z^4)$.

But by the preceding example, the factors of $x^4 - z^4$, are $(x^2 + z^2)$ $(x + z)$ $(x - z)$. These three factors written in place of their product in the first result, give us $(x^4 + z^4)$ $(x^2 + z^2)$ $(x + z)$ $(x - z)$.

13. Factor $m^{16} - c^{16}$.

Operating as before $(m^8 + c^8)$ $(m^8 - c^8)$. (1)

But by Ex. 12 $(m^8 - c^8)$ is equal $(m^4 + c^4)$ $(m^2 + c^2)$ $(m + c)$ $(m - c)$. Whence we have $(m^8 + c^8)$ $(m^4 + c^4)$ $(m^2 + c^2)$ $(m + c)$ $(m - c)$.

14. Factor $c^{32} - 1$.

Reasoning as in Ex. 13, we perceive the result at once.

15, 16, and 17. The operations and results are equally obvious.

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## FRACTIONS.

### (**109**, page 80.

6. Reduce $\dfrac{51a^3b - 63a^2b^2}{36a^4b^2 - 9ab}$ to its lowest terms.

By comparing $9ab$, the smallest term, with the other terms, we perceive that 3 is the greatest numeral divisor common to all the terms, and $ab$ is continued in all the terms. Therefore, $3ab$ is the greatest common factor, and dividing by it, we obtain $\dfrac{17a^2 - 21ab}{12a^3b - 3}$, *Ans.*

7. Reduce, &c., $\dfrac{4a^2-4x^2}{3(a+x)}$.

The factors of the numerator are readily determined by inspection; thus, $4(a^2-x^2)$ which is $4(a+x)\,(a-x)$, and omitting the factor $(a+x)$, common to both terms of the fraction, we obtain $\dfrac{4(a-x)}{3}$ for the reduced fraction.

8. Reduce, &c., $\dfrac{x^5-b^2x^3}{x^4-b^4}$

$$x^5-b^2x^3=(x^2-b^2)x^3$$
$$x^4-b^4=(x^2-b^2)\,(x^2+b^2)$$

Omitting the common factor $(x^2-b^2)$ and we have

$$\frac{x^3}{x^2+b^2}, \quad \textit{Ans.}$$

9. Reduce $\dfrac{x^2-1}{xy+y}$ to its lowest terms.

$$x^2-1=(x+1)\,(x-1)$$
$$xy+y=(x+1)y$$

Whence $\dfrac{x-1}{y}$ is the reduced fraction.

10. Reduce $\dfrac{cx+cx^2}{acx+abx}$ to, &c.

The only factor common to all the terms is $x$. Therefore, divide by $x$, and we obtain $\dfrac{c+cx}{ac+ab}$, *Ans.*

11. Divide $x^3y^2+x^2y^3$ by $a(x^2y+xy^2)$. Divide first by $x^2y+xy^2$, and that quotient by $a$; thus,

$$x^2y+xy^2)x^3y^2+x^2y^3(xy \qquad \text{whence } \frac{xy}{a}, \quad \textit{Ans.}$$
$$x^3y^2+x^2y^3$$

12. Divide $4(a+b)$ by $2(a^2-b^2)$

$$Ans.\ \frac{2(a+b)}{a^2-b^2}=\frac{2}{a-b}.$$

13. Divide $n^3-2n^2$ by $n^2-4n+4$:

Or, $\dfrac{(n-2)n^2}{(n-2)\,(n-2)}=\dfrac{n^2}{n-2}$, *Ans.*

14. $\dfrac{5a^2+5ax}{a^2-x^2}=\dfrac{5a(a+x)}{(a-x)\,(a+x)}=\dfrac{5a}{a-x}$, *Ans.*

15. Reduce $\dfrac{x^3-c^2x}{x^2+2cx+c^2}=\dfrac{(x^2-c^2)x}{(x+c)\,(x+c)}$

$\dfrac{(x+c)\,(x-c)x}{(x+c)\,(x+c)}=\dfrac{(x-c)x}{x+c}$, *Ans.*

16. Reduce $\dfrac{(x^2-a^2)x}{x^3-a^3}=\dfrac{(x+a)\,(x-a)x}{(x^2+ax+a^2)\,(x-a)}$

$\dfrac{(x+a)x}{x^2+ax+a^2}$, *Ans.*

17. $\dfrac{a^2x^4-a^2y^4}{x^4+x^2y^2}=\dfrac{a^2(x^4-y^4)}{x^2(x^2+y^2)}=\dfrac{a^2(x^2-y^2)}{x^2}$, *Ans.*

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## ADDITION.

(**116**, page 91.)

8. The fractions reduced to a common denominator are $\dfrac{a-b}{a^2-b^2}$ and $\dfrac{a+b}{a^2-b^2}$, and the sum is $\dfrac{2a}{a^2-b^2}$, *Ans.*

9. When reduced to a common denominator, the fractions will be $\dfrac{x^2-xy}{x^2-y^2}$ and $\dfrac{xy+y^2}{x^2-y^2}$. Whence their sum is $\dfrac{x^2+y^2}{x^2-y^2}$, *Ans.*

10 The denominators will be common, provided we multiply the terms of the second fraction by 5. Then we shall have $\frac{12b-a}{35c}+\frac{15a-5b}{35c}$. Sum is $\frac{14a+7b}{35c}=\frac{2a+b}{5c}$, *Ans.*

11. The first and last fractions have a common denominator; hence we add their numerators at once, making $\frac{1+a}{1+a}$; but this is 1 in value. Hence, the whole sum must be $1+\frac{a}{1-a}$, or $\frac{1-a+a}{1-a}=\frac{1}{1-a}$, *Ans.*

12. Adding the first two fractions, $\frac{3a}{3b}+\frac{a}{3b}=\frac{4a}{3b}$. To this we are to add $\frac{5b}{4a}$. Multiplying each numerator by the denominator of the other fraction, and placing the sum of the products over, &c., we obtain $\frac{16a^2+15b^2}{12ab}$ for their sum.

13. Multiply the second fraction by $4c$, and we have $\frac{12ac-16bc}{12bc}$. Adding the two fractions, and we have $\frac{6ab-3b^2}{12bc}$. Reducing, and we have $\frac{2a-b}{4c}$, *Ans.*

14. We add the entire quantities into one sum, $6x$, and then add the fractions $\frac{3a}{5}$ and $\frac{2a}{9}$ as a distinct example. The sum of the two results, $6x+\frac{37a}{45}$, is the result sought.

15. $5x+4x$ is $9x$. $\frac{x-2}{3}$ is equal $\frac{5x^2-10x}{15x}$, and $\frac{2x-3}{5x}=\frac{6x-9}{15x}$. Whence the whole sum is $9x+\frac{5x^2-4x-9}{15x}$.

(91, 92)

16. Multiply the second fraction by $(a - b)$. Then, the first fraction being $\frac{2b}{(a-b)(a+b)}$, and the second, $\frac{a-b}{(a-b)(a+b)}$, their sum must be $\frac{a+b}{(a-b)(a+b)}$, which reduced, is $\frac{1}{a-b}$, *Ans.*

17. Multiply the terms of the first fraction by $c$, of the second fraction by $a$, and of the third by $b$; then we shall have $\frac{ac-bc}{abc}$, $\frac{ab-ac}{abc}$, and $\frac{bc-ab}{abc}$, the sum of the numerators being 0, the value of the whole is 0, *Ans.*

18. Multiply the terms of the second fraction by $a$. Then we shall have $\frac{a^2-x^2}{ax}$, $\frac{ax-a^2}{ax}$. The sum of the two is $\frac{ax-x^2}{ax}$. Reduced, $\frac{a-x}{a}$, *Ans.*

19. Multiply the terms of the first fraction by $a$, and then add the first and second, and we shall have $\frac{5a+3}{ay}$ and $\frac{b}{3a}$, or $\frac{15a+9}{3ay}$ and $\frac{by}{3ay}$, to be added. Sum, $\frac{15a+by+9}{3ay}$, *Ans.*

21. Multiply the terms of the first fraction by $cd$, and of the second by $b$. Then we shall have $acd$, $ab - 3b^2$, and $a^2 - b^2 - ab$, for the three numerators. Their sum is $\frac{acd - 4b^2 + a^2}{bcd}$, *Ans.*

22. $\frac{a(a-b)}{(a+b)(a-b)}+\frac{(a+b)b}{(a+b)(a-b)}$, or $\frac{a^2-ab}{a^2-b^2}+\frac{ab+b^2}{a^2-b^2}=\frac{a^2+b^2}{a^2-b^2}$, *Ans.*

23. $\frac{x-y}{(x+y)(x-y)}+\frac{y}{x^2-y^2}=\frac{x}{x^2-y^2}$, *Ans*.

24. Multiply the terms of the second fraction by $1-a^2$, and it becomes $\frac{1-2a^2+a^4}{1-a^4}$. To this add $\frac{4a^2}{1-a^4}$, and the sum is $\frac{1+2a^2+a^4}{1-a^4}$. But $\frac{1+2a^2+a^4}{1-a^4}=\frac{(1+a^2)(1+a^2)}{(1+a^2)(1-a^2)}=\frac{1+a^2}{1-a^2}$, *Ans.*

25. The sum of $\frac{1}{a}+\frac{1}{b}$ is $\frac{a+b}{ab}$, and this sum added to the third quantity, makes the whole sum 1, *Ans.*

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## SUBTRACTION.

(**117**, page 94.)

6. Reducing the fractions to a common denominator, and we have $\frac{a^2-a+1}{(a+1)(a^2-a+1)}-\frac{a^2-a-2}{(a+1)(a^2-a+1)}$, which is $\frac{3}{(a+1)(a^2-a+1)}=\frac{3}{a^3+1}$, *Ans.*

9. $\frac{x+1}{x^2-1}-\frac{2x-2}{x^2-1}=\frac{3-x}{x^2-1}$, *Ans.*

10. The difference between the entire parts is $2x$. $\frac{ax-x^2}{ax}-\frac{ax-a^2}{ax}=\frac{a^2-x^2}{ax}$. *Ans.* $2x+\frac{a^2-x^2}{ax}$

11. $\dfrac{2a+b}{5c} \times \dfrac{7}{7} = \dfrac{14a+7b}{35c}$; $\dfrac{3a-b}{7c} \times \dfrac{5}{5} = \dfrac{15a-5b}{35c}$;
Diff. is $\dfrac{12b-a}{35c}$, *Ans.*

12. $\dfrac{20x+4}{28} - \dfrac{147x+21}{28} = -\dfrac{127x+17}{28}$, *Ans.*

13. From $\dfrac{3x-3y}{6a}$ take $\dfrac{2x+2y}{6a}$. Diff. $= \dfrac{x-5y}{6a}$, *Ans.*

14. From $\dfrac{(1+a^2)^2}{(1-a^2)(1+a^2)}$ take $\dfrac{(1-a^2)^2}{(1-a^2)(1+a^2)}$.
Diff. $\dfrac{4a^2}{1-a^4}$, *Ans.*

15. From $\dfrac{x-y}{x(x+y)}$ take $\dfrac{x+y}{x(x-y)}$. $\dfrac{x^2-2xy+y^2}{x(x^2-y^2)} - \dfrac{x^2+2xy+y^2}{x(x^2-y^2)}$. Diff. $-\dfrac{4xy}{x(x^2-y^2)} = -\dfrac{4y}{x^2-y^2}$. Whence, $x - \dfrac{4y}{x^2-y^2}$, *Ans.*

16. $\dfrac{(a-b)5d}{2c5d}$ less $\dfrac{(2b-4a)2c}{5d2c}$. Or,

$$\frac{5ad-5db-4bc+8ac}{10cd}, \textit{ Ans.}$$

17. From $\dfrac{2a^2+2b^2}{a^2-b^2}$ take $\left(\dfrac{a-b}{a+b}\right)\left(\dfrac{a-b}{a-b}\right)$. That is,

From $2a^2 + \quad 2b^2$
Take $\quad a^2 - 2ab + b^2$

Diff. $\dfrac{a^2+2ab+b^2}{a^2-b^2} = \dfrac{(a+b)(a+b)}{(a+b)(a-b)} = \dfrac{a+b}{a-b}$, *Ans.*

18. From $\dfrac{x^2}{x(x-3)}$ take $\dfrac{x^2-9}{x(x-3)}$. Diff. $\dfrac{9}{x(x-3)} = \dfrac{9}{x^2-3x}$, *Ans.*

19. From $6a$ take $4a$, is $2a$. And from $\frac{14a-13}{20}$ take $\frac{10a-25}{20}$. Diff. is $\frac{4a+12}{20}=\frac{a+3}{5}$. Whence

$$2a+\frac{a+3}{5}, \text{ Ans.}$$

20. From $\frac{a^2+b^2}{a^2-b^2}$ take $\frac{b(a+b)}{(a-b)(a+b)}$. That is, from $\frac{a^2+b^2}{a^2-b^2}$ take $\frac{ab+b^2}{a^2-b^2}=\frac{a^2-ab}{a^2-b^2}=\frac{a(a-b)}{(a+b)(a-b)}=$

$$\frac{a}{a+b}, \text{ Ans.}$$

21. From $\frac{1+2x^2+x^4}{(1-x^2)(1+x^2)}$ take $\frac{1-2x^2+x^4}{(1-x^2)(1+x^2)}$.

$$\text{Diff. } \frac{4x^2}{1-x^4}, \text{ Ans.}$$

22. From $\frac{ax-x^2}{ax}$ take $\frac{a^2-x^2}{ax}$. Diff. $\frac{ax-a^2}{ax}=$

$$\frac{x-a}{x}, \text{ Ans.}$$

23. Multiply the terms of the first fraction by $a$, the second by $b$, and the third by $c$. Then, from $\frac{ab+ac}{abc}+\frac{bc-ab}{abc}$ take $\frac{ac-bc}{abc}$. That is, from $\frac{ac+bc}{abc}$ take $\frac{ac-bc}{abc}$.

$$\text{Diff. } \frac{2bc}{abc}=\frac{2}{a}, \text{ Ans.}$$

24. Unite the first and third, $\frac{6x}{8}-\frac{5x}{8}=\frac{x}{8}$; to this add $\frac{2x}{5}$. $\frac{21x}{40}$, *Ans.*

25. $\frac{n^2-2n+1}{n^2-n}-\frac{n^2}{n^2-n}$. Diff. $\frac{1-2n}{n^2-n}$, *Ans.*

26. Multiply the terms of the subtrahend by $1 + x$. Then,
$\frac{2}{1-x^2} - \frac{1+x}{1-x^2} = \frac{1-x}{1-x^2} = \frac{1}{1+x}$, *Ans.*

27. Multiply the terms of the first fraction by $(x-b)$, and the terms of the second fraction by $(x-a)$. Then, from $\frac{a(x-b)+c(x-b)}{(a-b)(x-a)(x-b)}$ we must take $\frac{b(x-a)+c(x-a)}{(a-b)(x-a)(x-b)}$.

From $ax - ab + cx - cb$

Take $bx - ab + cx - ca$

Diff. $(a-b)x + (a-b)c$; or $\frac{(a-b)x+(a-b)c}{(a-b)(x-a)(x-b)}$.

Dividing the terms by $(a-b)$, and we have

$$\frac{x+c}{(x-a)(x-b)}, \textit{ Ans.}$$

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## MULTIPLICATION.

(**120**, page 100.)

18. Multiply $b + \frac{bx}{a}$ by $\frac{a}{x}$. That is, $\frac{ab+bx}{a} \times \frac{a}{x} =$

$$\frac{ab+bx}{x}, \textit{ Ans.}$$

19. Multiply $\frac{x^2-b^2}{bc}$ by $\frac{x^2+b^2}{b+c}$. Here no reductions can be made. Hence, $\frac{x^4-b^4}{b^2c+bc^2}$, *Ans.*

20. Multiply $\frac{a^2-x^2}{2y}$ by $\frac{2a}{a+x}$. That is,
$\frac{(a-x)(a+x)}{2y} \times \frac{2a}{(a+x)}$. By cancellation, $\frac{(a-x)a}{y}$, *Ans.*

21. By cancellation, we find $a$ for the result.

22. Multiply $\frac{3a}{1}, \frac{x+1}{2a}, \frac{x-1}{a+b}$. Omitting the common factor $a$, and the product of the other factors is

$$\frac{3(x^2-1)}{2(a+b)}, \textit{ Ans.}$$

23. Multiply $\frac{3x^2-5x}{14}$ by $\frac{7a}{2x^3-3x}$. Here 7 is a factor in 14, and $x$ is a factor common to a numerator and a denominator. Therefore, $\frac{3x-5}{2} \times \frac{a}{2x^2-3} = \frac{3ax-5a}{4x^2-6}$, *Ans.*

24. Multiply $\frac{3x^3}{5x-10}$ by $\frac{15x-30}{2x}$. That is, $\frac{3x^3}{5(x-2)} \times \frac{15(x-2)}{2x}$. Or, $\frac{3x^3.15}{5.2x} = \frac{3x.3}{2} = \frac{9x}{2}$, *Ans.*

NOTE. As a period, placed between algebraic quantities, denotes multiplication, we can use it instead of $\times$, where it is not liable to be mistaken for a decimal point.

26. Multiply $\frac{x^2-y^2}{ab}$ by $\frac{a^2}{x+y}$. That is, $\frac{(x-y)(x+y)}{ab} \times \frac{a^2}{(x+y)}$. Omitting common factors, and $\frac{a(x-y)}{b}$, *Ans.*

29. Multiply $\frac{(a-x)^3}{2a}, \frac{3ab}{a-x}, \frac{2c}{(a-x)^2}$. The product, $(a-x)^3$, of the second and third denominators, cancels the first numerator. Also, the factors 2 and $a$ will cancel. Then, $3b \times c = 3bc$, *Ans.*

30. By cancellation, we at once obtain $\frac{a+b}{x+y}$, *Ans.*

DIVISION.

(**122**, page 104.)

6. $\frac{15ab}{a-x} \times \frac{(a-x)(a+x)}{10ac}$. Cancelling the common factors $5a$ and $(a-x)$, and we have $\frac{3b(a+x)}{2c}$, *Ans.*

7. $\frac{my+x}{y} \times \frac{d}{c}$. Prod. is $\frac{d(my+x)}{cy}$, *Ans.*

8. $\frac{ax+ac}{x} \times \frac{x^2}{ac}$. Cancel the common factors $a$ and $x$, and we have $(x+c) \times \frac{x}{c} = \frac{x^2}{c} + x$, *Ans.*

9. Inverting the terms of the divisor, and we have $\frac{2ax+x^2}{a^2-x^2} \times \frac{a-x}{x}$. Cancelling the common factors $x$ and $(a-x)$, gives us $\frac{2a+x}{a^2+ax+x^2}$. *Ans.*

In this example we see that the terms of the divisor are factors of the corresponding terms of the dividend. Whenever this is the case, we may *divide numerator by numerator, and denominator by denominator.*

10. $\frac{14x-3}{5} \times \frac{25}{10x-4}$, which reduces to $\frac{(14x-3)5}{10x-4}$.

$$\frac{70x-15}{10x-4}, \text{ Ans.}$$

11. $\frac{9x^2-3x}{5} \times \frac{5}{x^2}$. Reduces to $\frac{9x-3}{x}$, *Ans.*

12. $\frac{6x-7}{x-1} \times \frac{3}{x+1} = \frac{18x-21}{x^2-1}$, *Ans.*

13. $\frac{16ax}{5} \times \frac{15}{4x}$. By reducing, $4a \times 3 = 12a$, *Ans.*

14. $\frac{6z+4}{5} \times \frac{4y}{3z+2}$. Reducing, $\frac{2 \times 4y}{5} = \frac{8y}{5}$, *Ans.*

15. $\frac{7x}{3} \times \frac{6}{4x^2}$. Reduces to $\frac{7}{2x}$, *Ans.*

16. $\frac{a+1}{6} \times \frac{3}{2a}$. Reduces to $\frac{a+1}{4a}$, *Ans.*

18. $\frac{(x-y)(x-y)}{ab} \times \frac{bc}{x-y}$. Reduces to

$\frac{(x-y)c}{a}$, *Ans.*

19. $\frac{(m+n)(m-n)}{3} \times \frac{6}{m-n}$. Reduces to

$2(m-n)$, *Ans.*

22. $\frac{(x^2+b^2)(x^2-b^2)}{(x-b)(x-b)} \times \frac{x-b}{x(x+b)}$. By cancellation,

$\frac{x^2+b^2}{x}$, which reduced to a mixed quantity, is $x + \frac{b^2}{x}$, *Ans.*

23. $\frac{a+1}{a} \times \frac{a^2}{a^2-1}$, or $\frac{a+1}{1} \times \frac{a}{(a+1)(a-1)}$.

Reduces to $\frac{a}{a-1}$, *Ans.*

24. $\left(\frac{1}{1+x} + \frac{x}{1-x}\right)$ divided by $\frac{1}{1+x} + \frac{x}{1+x}$. But this divisor is 1. Therefore, the quotient is $\frac{1}{1+x} + \frac{x}{1-x}$,

or $\frac{1-x+x+x^2}{1-x^2} = \frac{1+x^2}{1-x^2}$, *Ans.*

25. Multiply both dividend and divisor by $ab$. This will not affect the quotient. Then we shall have $b+\frac{1}{b^2}$ to divide by $a(b^2+1-b)$. Multiply each again by $b^2$, and we shall have $b^3+1$ to divide by $ab^2(b^2+1-b)$.

$$\begin{array}{l} b^2-b+1\,)\,b^3+1\,(\,b+ \\ \qquad\qquad\quad \underline{b^3-b^2+b} \\ \qquad\qquad\qquad\; b^2-b+1 \end{array}$$

Then, $\frac{b+1}{ab^2}$, *Ans.*

## (122.)

Division in fractions is sometimes more easily effected by placing the divisor under the dividend, thus making a complex fraction, which can be reduced and simplified.

Thus, in place of requiring the division of $a+\frac{m}{n}$ by $b-\frac{c}{d}$, we say simplify the fraction.

3. $\frac{a+\frac{m}{n}}{b-\frac{c}{d}}$. Multiply both terms by $nd$.

Then, $\frac{adn+dm}{bdn-cn}$, *Ans.*

4. Reduce $\frac{a+\frac{a}{3}}{b}$ to, &c. $\frac{a+\frac{a}{3}}{b}\times\frac{3}{3}=\frac{4a}{3b}$, *Ans.*

5. Simplify $\frac{\frac{1}{4}a+c}{x+\frac{1}{2}z}$. $\frac{\frac{1}{4}a+c}{x+\frac{1}{2}z}\times\frac{4}{4}=\frac{a+4c}{4x+2z}$, *Ans.*

6. Simplify $\frac{m}{a+\frac{1}{c}}$. Multiply both terms by $c$.

8. Multiply both terms by $ax^2$.

## (105, 106)

9. Multiply both terms by $2x$. Then,

$$\frac{10cx + a - b}{10cx - a + b}, \textit{ Ans.}$$

10. Simplify $\dfrac{\dfrac{m^2}{m^2 - n^2} - 1}{\dfrac{n^2}{m^2 - n^2} + 1}$. Multiply by $m^2 - n^2$, and

$$\frac{m^2 - m^2 + n^2}{n^2 + m^2 - n^2} = \frac{n^2}{n^2}, \textit{ Ans.}$$

11. Simplify $\dfrac{\dfrac{a^3 - x}{2b^4}}{\dfrac{c+1}{2bx^2y^3}} \times \dfrac{2b^4}{2b^4} = \dfrac{a^3 - x}{\dfrac{(c+1)b^3}{x^2y^3}} \times \dfrac{x^2y^3}{x^2y^3} =$

$$\frac{(a^3 - x)x^2y^3}{b^3(c+1)}, \textit{ Ans.}$$

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## SIMPLE EQUATIONS.

(**140**, page 116.)

12. Multiply as indicated, and we obtain

$$3x + 3 + 4x + 8 = 6x + 18.$$

Omit $6x$ and 11 on each side, and $x = 7$, *Ans.*

13. Multiply, and $5x + 5 + 6x + 12 = 6x + 42$.

$$5x = 25;\ x = 5, \textit{ Ans.}$$

14. Multiply, and $7x + 21 - 12x + 64 = 45$.

$$85 - 45 - 5x = 0;\ 40 - 5x = 0;\ x = 8.$$

16. Factor each member.

Then $(c-1)x=(c-1)b$; whence $x=b$.

17. Factor, and

$(a+d)x=a-c$; whence $x=$ *Ans.*

18. By transposition, $ax-cx=n-m$.

Factoring, $(a-c)x=n-m$; whence $x=$ *Ans.*

19. By transposition and factoring,

$(a-b-d)x=c-m$; whence $x=$ *Ans.*

23. Multiply by 24, the least common multiple of the denominators, to clear of fractions. Then,

$6x+3x-4x=10$; or, $5x=10$; $x=2$, *Ans.*

24. Multiply by 12, and we obtain $\frac{15x}{2}+3=22+7x$.

Multiply by 2, and $15x+6=44+14x$; whence $x=38$, *Ans.*

25. Omit $2b$ from each member, and multiply by $2a$, and we shall obtain

$$2x+ax-5a=ab;\text{ whence } x=\frac{ab+5a}{2+a}.$$

26. Omitting 13 in each member, and then

$$\frac{3x}{5}+\frac{1}{2}=\frac{x}{4}+4.$$

Multiply by 20, and $12x+10=5x+80$; $7x=70$; and $x=10$, *Ans.*

28. Multiply every term by 60. Then,

$$30x + 20x + 15x + 12x = 77 \times 60.$$

$$\text{Or, } 77x = 77 \times 60.$$

Dividing by 77, and $x = 1 \times 60$, or $x = 60$, *Ans.*

29. Multiply every term by 12. Then,

$$6x + 4x + 3x = 130 \times 12.$$

$$13x = 130 \times 12; \text{ or, } x = 10 \times 12 = 120, \textit{ Ans.}$$

30, 31, 32. Operations similar to that in Ex. 29.

34. By transposing the terms having the minus sign, we shall have

$$x + \frac{x+4}{3} = \frac{9}{2} + \frac{x-3}{2}.$$

Multiply by 6, and $6x + 2x + 8 = 27 + 3x - 9.$

$$5x = 10; \text{ and } x = 2, \textit{ Ans.}$$

35. Transposing as in the preceding, we obtain

$$\frac{x+2}{3} + \frac{x-1}{2} + 2 = x + \frac{x-3}{4}$$

Multiplying by 12, and

$$4x + 8 + 6x - 6 + 24 = 12x + 3x - 9.$$

$$-5x = -35, \text{ and}$$

$$x = 7, \textit{ Ans.}$$

36. Transpose and multiply by 11, all in one operation, and we have

$$4x - 2 = \frac{33x - 55}{13} + 11.$$

Multiply by 13, and $52x - 26 = 33x - 55 + 143.$

Dropping $33x$ and $-26$ from each member, and

$$19x = -29 + 143 = 114.$$

$$x = 6, \textit{ Ans.}$$

(118, 119)

37. By transposing,

$$\frac{x}{5}+\frac{x}{2}=\frac{x-2}{3}+\frac{13}{3}.$$

Multiply by 30, and

$$6x+15x=10x-20+130.$$

$$\text{Or, } 11x=110;\ x=10, \textit{ Ans.}$$

38. Each term multiplied by 12, gives

$$6x+4x+3x=12a.$$

$$13x=12a.$$

$$x=\frac{12a}{13}, \textit{ Ans.}$$

39. Each term multiplied by $abc$, and

$$bcx+acx+abx=abc.$$

$$\text{Or, } (bc+ac+ab)x=abc.$$

$$x=\frac{abc}{bc+ac+ab}, \textit{ Ans.}$$

40. By clearing of fractions, we have

$$a+ax=1+a-x-ax.$$

$$2ax+x=1.$$

$$\text{Or, } (2a+1)x=1.$$

$$x=\frac{1}{2a+1}, \textit{ Ans.}$$

41. By transposing the minus terms, we obtain

$$\frac{x+3}{4}+1=\frac{x-8}{5}+\frac{x-5}{2}.$$

$$5x+15+20=4x-32+10x-50.$$

$$-9x=-117.$$

$$x=13, \textit{ Ans.}$$

42. By transposing and uniting the numerals, we obtain

$$\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=94.$$

Multiplying by 60, and

$$20x+15x+12x=94\times 60.$$

$$\text{Or, } 47x=94\ .\ 60.$$

$$x=2\ .\ 60=120, \textit{ Ans.}$$

43. Multiplying by 3, we obtain

$$\frac{45}{x+3}-\frac{1}{x+3}=6\tfrac{14}{5}=8\tfrac{4}{5}.$$

Multiply by 5, $\dfrac{225}{x+3}-\dfrac{5}{x+3}=44.$

Multiply by $(x+3)$, and

$$225-5=44x+132.$$

$$220=44x+132.$$

$$88=44x;\ \text{or, } x=2, \textit{ Ans.}$$

45. Divide by the product of the two known factors, and we obtain

$$x-m=\frac{b^2m}{a^2-b^2}.$$

Whence $x=m+\dfrac{b^2m}{a^2-b^2}=\dfrac{a^2m-b^2m+b^2m}{a^2-b^2}=\dfrac{a^2m}{a^2-b^2}$, *Ans.*

46. Multiply each term by 3, and transpose the minus fraction, and

$$\frac{11x-80}{2}=\frac{8x-5}{5}.$$

$$55x-400=16x-10.$$

$$39x=390;\ x=10, \textit{ Ans.}$$

47. Multiply each member by $\frac{x}{6}$, and we shall have

$$1 + \frac{x}{4} + m = m + 6;\ \text{whence}\ \frac{x}{4} = 5;\ \text{or},\ x = 20,\ \textit{Ans.}$$

49. Assume $x + 5 = y$. Then,

$$\frac{y}{2} + \frac{5y}{6} = 3y - 20.$$

Multiply by 6, and $3y + 5y = 18y - 120$.

$$\text{Or},\ 10y - 120 = 0;\ y = 12.$$

But $x + 5 = y$; that is, $x + 5 = 12$; whence $x = 7$, *Ans.*

50. Place $x - a = y$. Then,

$$y + \tfrac{1}{2}y + \tfrac{3}{4}y = \tfrac{9}{4}.$$

Multiply by 4, and $4y + 2y + 3y = 9$.

$$\text{Or},\ 9y = 9;\ y = 1.$$

Whence $x - a = 1$; and $x = a + 1$, *Ans.*

53. Multiply each term by 21. Then we shall have

$$7x + 16 = x + 8 + 7x.$$

Dropping $7x + 8$ from each member, and we have $8 = x$, *Ans.*

54. Multiply each term by 21. Then,

$$7x + 2a = \frac{21x + 21a}{4x - 11} + 7x.$$

Dropping $7x$ from each member, and

$$2a = \frac{21x + 21a}{4x - 11}.$$

$$\text{Whence}\ 8ax - 22a = 21x + 21a.$$

$$(8a - 21)x = 43a.$$

$$x = \frac{43a}{8a - 21}.$$

NOTE. If we make $8a - 21 = 43$. Then, $8a = 64$; $a = 8$; $x = a$.

55. Transpose the minus quantities, and we have

$$\frac{2x+1}{29}+\frac{471-6x}{2}=9+\frac{402-3x}{12}.$$

Multiply each term by 2, and divide the last numerator by 6; then we shall have

$$\frac{4x+2}{29}+471-6x=18+67-\tfrac{1}{2}x.$$

By uniting the numbers, and transposing $6x$, we obtain

$$\frac{4x+2}{29}+386=6x-\tfrac{1}{2}x.$$

Multiply by 2, and $\dfrac{8x+4}{29}+772=11x.$

Clearing of fractions, and $8x+4+22388=319x.$

Whence $311x=22392.$

By division, $x=72$, *Ans.*

56. Multiply each term by 2; that is, divide each denominator mentally by 2, as we write it, and we will have

$$\frac{18x-19}{14}+\frac{11x+21}{3x+7}=\frac{9x+15}{7}.$$

Multiply each term by 14. Then,

$$18x-19+\frac{14(11x+21)}{3x+7}=18x+30.$$

Omit $18x$, and transpose $-19$, and

$$\frac{14(11x+21)}{3x+7}=49.$$

Divide by 7, and $\dfrac{2(11x+21)}{3x+7}=7$

Clearing of fractions, $22x+42=21x+49.$

Whence $x=7.$

(121, 122)

57. Divide the numerator of the second member by 3, as indicated. Then we shall have

$$\frac{1}{x-5}+x+5=x+6.$$

By dropping $(x+5)$ from both members, and $\frac{1}{x-5}=1.$

Whence $1=x-5$; or, $6=x$, *Ans.*

58. Multiply each term by $x$, mentally. Then,

$$\frac{mx(3a^2-2b^2)}{a+b}=\frac{2a^2-b^2}{a+b}+a-b.$$

Multiply by $(a+b)$, and

$$mx(3a^2-2b^2)=2a^2-b^2+a^2-b^2=3a^2-2b^2.$$

Divide by $(3a^2-2b^2)$, and

$$mx=1;\ \text{whence } x=\frac{1}{m},\ \textit{Ans.}$$

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PROPORTION.

(141, page 123.)

5. Given $x+6 : 38-x :: 9 : 2$, to find $x$.
Multiply extremes and means,

$$2x+12=342-9x.$$

By transposition, $11x=330$; and $x=30$, *Ans.*

6. Given $x+4 : x-11 :: 10 : 4$, to find $x$.
Prod. &c., $4x+16=10x-110.$

$$-6x=-126;\ \text{whence } x=21,\ \textit{Ans.}$$

7. Given $x + a : x - a :: c : d$, to find $x$.

Prod. &c., $dx + ad = cx - ac$.

By transposition, $ad + ac = (c - d)x$

Or, $(c - d)x = a(c + d)$.

Whence $x = \frac{a(c + d)}{c - d}$, *Ans.*

(Page 124.)

8. Given $x : 2x - a :: a : b$, to find $x$.

$$2ax - a^2 = bx.$$

$$(2a - b)x = a^2.$$

Whence $x = \frac{a^2}{2a - b}$, *Ans*

10. Divide, mentally, the 1st and 2d terms by $a$. Then,

$$a - c : x :: 1 : (d - b).$$

Multiply extremes, and $x = (d - b)(a - c)$, *Ans.*

PROBLEMS PRODUCING EQUATIONS.

(Page 126.)

4. Let $x =$ the price of the harness;

Then $3x =$ " " chaise;

And $3x + 20 =$ " " horse.

*Sum*, $7x + 20 = 230$

Whence $7x = 210$

And $x = 30$, the price of the harness.

5. Let $x =$ the sum paid by one man;

And $5x + 26 =$ " " the other.

Both paid $6x + 26 = 86$;

$6x = 60$; and $x = 10$

Hence \$10 and \$76, *Ans.*

6. Let $x =$ the sum given to the youngest son;

Then $x + 4 =$ " to the next older;

$x + 8 =$ " " "

$x + 12 =$ " " "

$x + 16 =$ " " "

$x + 20 =$ " " eldest.

Sum, $6x + 60 = 120$;

$x + 10 = 20$; and $x = 10$

7. Let $x =$ the sum received by the first;

Then, $x + 5 =$ " " " second;

And $x + 15 =$ " " " third.

$3x + 20 = 65$;

$3x = 45$; $x = 15$, *Ans.*

8. Let $x =$ the first payment;

Then, $x + 3 =$ the second "

And $2x + 6 =$ the third "

$4x + 9 = 29$;

Whence $x = 5$, *Ans.*

9. Let $x =$ the less number;

Then, $x + 14 =$ the greater.

By the second condition, $10x = 3x + 3 \times 14$;

$7x = 3 \times 14$;

and $x = 3 \times 2 = 6$.

10. Let $x =$ the age of Moses;

And $x + 16 =$ " Joseph.

By the second condition, $5x = 3x + 3 \times 16$;

$2x = 3 \times 16$; and $x = 3 \times 8 = 24$

11. Let $x =$ the sum paid to A;

$x - 500 =$ " " B;

and $x + 900 =$ " " C;

Sum, $3x + 400 = 2500$;

$3x = 2100$; $x = 700$, *Ans.*

12. Let $x =$ the first number;

$2x =$ the second "

and $3x =$ the third "

$6x = 72$; and $x = 12$

13. Let $x =$ the sum paid to A;

and $4x =$ " " B.

$5x = 750$; and $x = 150$

(Page 128.)

15. Let $x =$ the value of the saddle;

and $nx =$ " " horse.

Sum, $(n + 1)x = a$;

Whence $x = \frac{a}{1+n}$, saddle; $\frac{na}{1+n}$, horse; *Ans.*

16. Let $x =$ the number of horses

$4x =$ " cows;

and $20x =$ " sheep.

Sum, $25x = 100$, by the last condition.

Whence $x = 4$, *Ans.*

17. This is the same as Problem 16, in a literal form.

Let $x =$ the number of horses;

$nx =$ " cows;

and $mnx =$ " sheep.

Sum, $(1 + n + mn)x = a$, by last condition.

Whence $$x = \frac{a}{1 + n + mn}$$

If $a = 100$, $n = 4$, and $m = 5$, then $x = 4$, the same as in the preceding problem.

18. Let $x =$ the number of needles;

and $7x =$ " pins.

Sum, $8x = 120$, by first condition.

Whence $x = 15$, needles; and $7x = 105$, pins; *Ans.*

19. Let $x =$ the number of scholars in Algebra;

$3x =$ " " Arithmetic;

and $12x =$ " " Grammar.

Sum, $16x = 64$; and $x = 4$

20. Let $x =$ the number in Algebra;

$nx =$ " Arithmetic;

and $mnx =$ " Grammar.

Sum, $(1 + n + mn)x = a$, by the first condition.

Whence $$x = \frac{a}{1 + n + mn}.$$

If $a = 64$, $n = 3$, and $m = 4$, then $x = 4$, as in Prob. 19.

21. Let $x =$ the sum due to A;

$2x =$ " " B;

and $6x =$ " " C.

Sum, $9x = 450$, the given condition.

Whence $x = 50$, the sum due to A.

22. Let $x =$ the sum due A;

$4x =$ " " B;

$8x =$ " " C;

and $6x =$ " " D.

$19x = 570$, total indebtedness.

Whence, $x = 30$, *Ans.*

23. Let $x =$ the sum due A;

$nx =$ " " B;

$mx =$ " " C;

and $px =$ " " D.

Then, $x + nx + mx + px = a$,

or, $(1 + n + m + p)x = a$.

Whence, $x = \dfrac{a}{1 + n + m + p}$.

**Note.** Any number of numerical problems like (20) can be made for this result, by giving to $n$, $m$ and $p$ different values, and making $a$ any multiple of $(1 + n + m + p)$.

25. Let $7x =$ one part;

and $8x =$ the other.

Then, $15x =$ the whole, or 150.

Hence, $x = 10$, and the two parts are 70 and 80, *Ans.*

26. Let $3x =$ A's share;

and $2x =$ B's share.

Whence, $5x = 1235$;

or, $x = 247$.

Therefore, $3x = 247 \times 3 = 741$, A's share;

and $2x = 494$, B's share.

27. Let $mx =$ A's share;

and $nx =$ B's share.

Then the two shares are to each other as $m$ to $n$.

Their sum is $mx + nx$, or $(m + n)x = d$;

whence, $x = \dfrac{d}{m+n}$.

Therefore, $mx = \dfrac{md}{m+n}$, A's share

28. Let $x + 40 =$ the dollars put in by A;

and $x =$ the number of dollars put in by B.

Then, by the condition, $x + 40 : x :: 5 : 4$

Whence, $4x + 160 = 5x$;

or, $x = 160$, the sum B put in.

29. Let $x =$ the value of the suit of clothes;

and $100 + x =$ " " 12 months' service.

Now, since $\frac{2}{3}$ of 12 months is 8 months, $\frac{2}{3}$ of $100 + x$, which is $\dfrac{200 + 2x}{3}$, must = the value of 8 months' service.

But this value is also equal to $60, and his suit to $x$.

Therefore, $\dfrac{200 + vx}{3} = 60 + x$;

whence, $200 + 2x = 180 + 3x$,

or, $x = 20$, *Ans.*

30. Let $2x =$ A's stock, $3x =$ B's stock, and $5x =$ C's stock.

Then, $10x = 780$, and $x = 78$.

Hence, $78 \times 2 = 156$, A's; $78 \times 3 = 234$, B's; and $390 = C's stock, *Ans.*

31. Let $5x =$ A's part; $11x =$ B's part, and $16x =$ C's part;

whence, $32x = 864$, and $x = 27$.

Therefore, the parts are, $27 \times 5$, $27 \times 11$, and $27 \times 16$;

or, A's part 135, B's 297, and C's 432.

32. Let $3x =$ the sum A put in, $7x =$ the sum B put in, and $5x =$ the sum C put in;

then they all put in $15x = 960$.

Double the equations, $30x = 1920$;

and $3x = 192 =$ the sum A put in.

33. Let $x =$ the number in each flock; then $x - 80$ and $x - 20$ are the numbers left.

By the conditions of the problem we have,

$$x - 80 : x - 20 :: 2 : 3$$

Whence, $3x - 240 = 2x - 40$;

and $x = 200$, *Ans.*

34. Let the required number of years be represented by $x$.

Then, $25 + x : 15 + x :: 5 : 4$

Whence, $100 + 4x = 75 + 5x$;

and $x = 25$, *Ans.*

35. Let $3x$ and $4x$ represent the numbers.

Then, $3x + 24 : 4x + 24 :: 4 : 5$

Whence, $15x + 5 \times 24 = 16x + 4 \times 24$,

and $1 \times 24 = x$, or $x = 24$.

Hence, the numbers are 72 and 96, *Ans.*

(130, 131)

36. Let $3x =$ the age of the husband, and $2x =$ the age of his wife. Then,

$$3x + 4 : 2x + 4 : : 7 : 5$$

Whence, $15x + 20 = 14x + 28$, and $x = 8$.

$8 \times 3 = 24$, and $8 \times 2 = 16$, *Ans.*

37. Let $x$ and $x + 12$ represent the numbers.

$$x + 12 : x : : 11 : 7.$$

$11x = 7x + 84$, $4x = 86$, and $x = 21$, the less number.

38. Let $x$ and $x + a$ represent the two numbers

Then, by the given conditions, we must have,

$$x + a : x : : m : n$$

Whence, $nx + na = mx$, or $(m - n)\,x = na$.

Dividing by $(m - n)$, and $x = \frac{na}{m - n} =$ one number.

The other number must be, $a + \frac{na}{m - n} = \frac{am}{m - n}$.

39. Let $x$ and $20 - x$ represent the numbers; then their difference is $20 - 2x$.

By the conditions we have the proportion,

$$20 : 20 - 2x : : 10 : 1$$

Whence, $20 = 200 - 20x$

$20x = 180$, or $x = 9$, one number, and $20 - 9 = 11$, the other.

40. Let $x$ and $a - x$ represent the numbers; then their difference is $a - 2x$.

$$a : a - 2x : : m : n$$

$$na = ma - 2mx$$

$2mx = (m - n)\,a$, and $x = \frac{(m - n)\,a}{2m}$, one number.

The other is, $a - \frac{(m - n)\,a}{2m}, \frac{2am - am + an}{2m} = \frac{(m + n)\,a}{2m}$.

42. Let $x =$ the principal, $r =$ the rate per cent.; $a = 113$; and $d = (120 - 113 =)\ 7$.

$$\text{Then, } \frac{rx}{12} = \text{the interest for one month,}$$

$$\frac{13rx}{12} = \text{the interest for 13 months,}$$

$$\text{and } \frac{20rx}{12} = \text{the interest for 20 months.}$$

Then, by the conditions, we obtain,

$$\frac{13rx}{12} + x = a, \qquad (1)$$

$$\text{and } \frac{20rx}{12} + x = a + d. \qquad (2)$$

Subtracting (1) from (2), and

$$\frac{7rx}{12} = d. \qquad (3)$$

Subtracting (3) from (1), and we obtain

$$\frac{rx}{2} + x = a - d, \qquad (4)$$

$$rx = 2a - 2d - 2x \quad (5)$$

From (1) $\quad 13rx = 12a - 12x \qquad (6)$

Dividing (6) by (5), and

$$13 = \frac{6a - 6x}{a - d - x},$$

$$13a - 13d - 13x = 6a - 6x,$$

$$7a - 13d - 7x = 0,$$

$$x = a - \frac{13d}{7}.$$

Restoring values, $x = 113 - \frac{13 \times 7}{7} = 100$, *Ans.*

The above is a natural mode of solution; but, by a little more thought, the solution can be much abridged, as follows:

When $a =$ the amount, and $x$ the principal, then $a - x$ must represent the interest for 13 months, and $a + d - x$ must be the interest for 20 months.

Now, whatever be the rate per cent., interest accumulates as the time; therefore,

$$a - x : a + d - x :: 13 : 20$$

$$13a + 13d - 13x = 20a - 20x$$

$$7x = 7a - 13d$$

$$x = a - \frac{13d}{7}, \textit{ Ans.}$$

43. Let $x$ represent the number. Then, by the conditions of the problem, we shall have

$$x - 45 : x + 45 :: 1 : 31$$

Whence, $31x - 31 \times 45 = x + 45$

By transposition, &c. $30x = 32 \times 45$,

$$10x = 32 \times 15,$$

$$5x = 16 \times 15;$$

$$x = 48, \textit{ Ans.}$$

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51. Let $x =$ the greater part; and $48 - x =$ the less.

And by the given condition, $\frac{x}{6} + \frac{48 - x}{4} = 9$.

Multiply by 12, and $2x + 144 - 3x = 108$

$$36 - x = 0;$$

and $x = 36$.

52. Let $x =$ his salary; then,

$\frac{x}{3} =$ what he had left after paying his board;

and $\frac{2}{3}$ of $\frac{x}{3} = \frac{2x}{9}$, the sum spent for clothes.

Then, $\frac{x}{3} - \frac{2x}{9} = 150$, and

$3x - 2x = 1350$, *Ans.*

*Another Solution.*—Let $9x =$ his salary; then, after paying board, he will have $3x$ left. Paying out $\frac{2}{3}$ of this, or $2x$, for clothes, and he will have $x$ left.

Whence, $x = 150$; and $150 \times 9 = 1350$, *Ans.*

53. Let $x =$ the whole estate;

then, $\frac{x}{4} + 200 =$ the share of the 1st child.

$\frac{x}{5} + 360 =$ " " 2d "

$\frac{x}{6} + 300 =$ " " 3d "

$\frac{x}{8} + 400 =$ " " 4th "

Sum, $\frac{x}{4} + \frac{x}{5} + \frac{x}{6} + \frac{x}{8} + 1240 = x$, the whole estate.

Multiply by 8, $2x + \frac{8x}{5} + \frac{4x}{3} + x + 1240 \times 8 = 8x.$

Dropping $3x$, $\frac{8x}{5} + \frac{4x}{3} + 1240 \times 8 = 5x.$

Multiply by 15, $24x + 20x + 1240 \times 120 = 75x.$

Transposing, &c. $1240 \times 120 = 31x.$

Dividing by 31, and $40 \times 120 = x$

or, $x = 4800$, *Ans.*

*Another Solution.*—The common multiple of 4, 5, 6, 8, is 120. Therefore we will let $120x$ represent the whole estate.

$$\begin{aligned} \text{Then, } 30x + 200 &= \text{share of 1st,} \\ 24x + 340 &= \quad \text{"} \quad \text{2d,} \\ 20x + 300 &= \quad \text{"} \quad \text{3d,} \\ 15x + 400 &= \quad \text{"} \quad \text{4th.} \\ \hline 89x + 1240 &= 120x, \\ 1240 &= 31x, \text{ or } x = 40; \end{aligned}$$

$$\text{Whence,} \qquad 1240 \times 40 = 4800, \textit{ Ans.}$$

54. $3 \times 5 \times 8 = 120$, a common denominator of all the given fractions.

$$\begin{aligned} \text{Let} \quad 120x &= \text{the number in the detachment;} \\ 80x &= \quad \text{"} \quad \text{on duty;} \\ \text{and} \quad 15x &= \quad \text{"} \quad \text{sick.} \end{aligned}$$

The remainder is $120x - 95x$, or $25x$; $\frac{1}{5}$ of this number, or $5x$, are on leave of absence. The remainder, 380, have deserted.

$$\begin{aligned} \text{Hence,} \quad 120x &= 80x + 15x + 5x + 380; \\ 20x &= 380; \text{ and } x = 19. \\ 120 \times 19 &= 2280, \textit{ Ans.} \end{aligned}$$

55. Let $x =$ the time past;

and $99 - x =$ " to come. $a = 99$.

By the condition, $\frac{3}{5}x = \frac{4a - 4x}{5}$;

$$\begin{aligned} 10x &= 12a - 12a \\ 22x &= 12a \\ 11x &= 6a; \text{ and } x = 6 \times 9 = 54, \textit{ Ans.} \end{aligned}$$

56. Let $x =$ the less part;

and $a - x =$ the greater.

By the conditions, $a - x - \frac{2x}{5} = 4x - \frac{3a - 3x}{7}$.

Multiply by 35, $35a - 35x - 14x = 140x - 15a + 15x$.

By transposition and reduction, $50a = 204x$.

But $a = 204$;

hence, $ax = 50a$;

and $x = 50$, the less part.

58. Let $4x =$ C's share;

$3x =$ B's "

$\frac{9x}{5} =$ A's "

Whence, $7x + \frac{9x}{5} = 44$.

$35x + 9x = 44 \times 5$,

$44x = 44 \times 5$,

and $x = 5$. $20 =$ C's share, &c.

61. The least common denominator of the given fractions is 12. Hence,

let $12x =$ the trees in the orchard.

Then, $6x =$ apple trees;

$3x =$ peach trees;

and $2x =$ plum trees.

Whence, $12x = 6x + 3x + 2x + 200$;

or, $x = 200$;

and $12 \times 200 = 2400$, *Ans.*

62. 24 is the least common denominator of the given fractions.

Let $24x =$ his whole number of sheep.

$$\begin{aligned}
6x &= \text{the number in the first field;}\\
4x &= \text{“ \quad “ \quad second field;}\\
3x &= \text{“ \quad “ \quad third field;}\\
2x &= \text{“ \quad “ \quad fourth field.}
\end{aligned}$$

Whence, $24x = 6x + 4x + 3x + 2x + 45.$

$$9x = 45;$$

$$x = 5; \text{ and } 24 \times 5 = 120, \textit{ Ans.}$$

63. Let $x =$ my money.

Then, by the required condition, we have

$$x - \frac{x}{4} - \frac{x}{5} = 66$$

Multiply by 12, and $20x - 3x - 4x = 66 \times 20,$

$$11x = 66 \times 20,$$

whence, $x = 6 \times 20 = 120$, *Ans.*

64. Let $x =$ my money.

Then,
$$x - \frac{x}{n} - \frac{x}{m} = a.$$

Multiply by $mn$, and $mnx - mx - nx = amn.$

$$(mn - m - n)x = amn.$$

Hence, $x = \dfrac{amn}{mn - m - n}.$

This problem is the preceding one in general terms.

65. Let $x =$ my height in inches.

Then, $\left(\frac{x}{3} - 12\right)\frac{1}{5} = 2;$

$$\frac{x}{3} - 12 = 10;$$

and $x = 66$ inches, or 5 feet 6 inches, *Ans.*

(135, 136)

66. Let $40x$ represent his fortune. Then, $15x$ was spent the 1st year, and $25x$ was left. The 2d year, $\frac{4}{5}$ of $25x$, or $20x$, was spent, and $5x$ only was left.

Hence, $5x = 1420$.

Multiply by 8, and $40x = 11360$, his fortune, *Ans.*

*Second Solution.*—Let $x$=his fortune. After the 1st year, $\frac{5x}{8}$ was left.

Now, $\frac{4}{5}$ of $\frac{5x}{8} = \frac{x}{2}$, the sum spent the 2d year.

Hence, $\frac{5x}{8} - \frac{x}{2} = 1420$.

Multiply by 8, and $5x - 4x = x = 11360$, *Ans.*

67. Let $12x$ = his money. He lost $3x$, and had $9x$ left. Then, from $9x + 3$, he lost $3x + 1$, and $6x + 2$ remained; and this was equal to 12. Whence, $6x = 10$, and $12x = 20$, *Ans.*

*Another Solution.*—Let $x$ = his money. After he lost, he had $\frac{3x}{4}$. Then he won 3, and from $\frac{3x}{4} + 3$, he lost $\frac{1}{3}$ of it, and had left $\frac{2x}{4} + 2 = 12$; or, $\frac{x}{2} = 10$, and $x = 20$, *Ans.*

68. Let $12x$ = his money. Then he lost $3x$, and had $9x$. From $9x + 3$, he lost $3x + 1$, and had remaining $6x + 2$, to which he won 2 shillings. He then had $6x + 4$. He lost $\frac{1}{7}$ of this, and had remaining $\frac{6}{7}$; that is, $\frac{6}{7}$ of $6x + 4$, which is 12.

$$(6x + 4)\tfrac{6}{7} = 12;$$

Dividing by $\frac{6}{7}$, $6x + 4 = 14$;

$$6x = 10, \text{ and } 12x = 20, \textit{ Ans.}$$

69. Let $x =$ the number of sheep;

Then, $\frac{1}{2}x - \frac{1}{2} =$ the number after the 1st robbery;

$x - \frac{1}{4} - \frac{1}{2} =$ the number after the 2d robbery;

$$\frac{x}{8} - \frac{1}{8} - \frac{1}{4} - \frac{1}{2} = 5$$

$x - 1 - 2 - 4 = 40$, and $x = 47$, *Ans.*

70. Let $8x =$ the price of the horse;

and $a - 8x =$ " " chaise.

$$2a - 16x - 3x = 24x - \frac{5a - 40x}{7}.$$

By transposing, $2a + \frac{5a - 40x}{7} = 43x.$

$$14a + 5a - 40x = 301x,$$

$$19a = 341x;$$

or, $ax = 19a$; $x = 19$; and $19 \times 8 = 152$, price of the horse, *Ans.*

72. Let $x =$ the number of days, as in 71.

Then, by working together they will do $\frac{1}{x}$ in one day. But A does $\frac{1}{a}$, and B, $\frac{1}{b}$, in one day.

Therefore, $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}.$

Whence, $x = \frac{ab}{a + b}$, *Ans*

73. By placing $a = 12$, and $b = 24$, these values substituted in 72 will give the answer to 73.

$$\frac{24 \times 12}{36} = 8, \textit{ Ans.}$$

74. Here, as in 72 and 73, let $x =$ the number of days required when all work together.

Then, $\frac{1}{5} + \frac{1}{6} + \frac{1}{8} = \frac{1}{x}$.

Whence, $\frac{x}{5} + \frac{x}{6} + \frac{x}{8} = 1$;

$48x + 40x + 30x = 240$;

$118x = 240$; and $x = 2\frac{2}{59}$, *Ans.*

76. Let $x =$ the number of idle days;

$a - x =$ " working days;

$ab - bx =$ the value of the labor;

and $cx =$ the amount of forfeiture.

Then, $ab - bx - cx = d$;

whence, $ab - d = (b + c)x$;

or, $x = \frac{ab - d}{b + c}$, *Ans.*

77. Let $x =$ the number he broke;

and $30 - x =$ " delivered safely.

By the conditions, we obtain

$150 - 5x - 12x = 99$.

Whence, $51 = 17x$, and $x = 3$, *Ans.*

*Another Solution.*—Let $x =$ the number he delivered safely.

and $30 - x =$ " broke.

Then, by the given conditions, we obtain the following equation:

$5x - 12.30 + 12x = 99$

$17x = 459$; $x = 27$; and $30 - 27 = 3$, *Ans.*

78. Let $x =$ the number he broke;

and $n - x =$ " delivered.

By the conditions, we obtain the following equation:

$$an - ax - bx = d.$$

Whence, $x = \dfrac{an - d}{a + b}$, *Ans.*

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## SIMPLE EQUATIONS

### CONTAINING TWO UNKNOWN QUANTITIES.

(**152**, page 146.)

9. Add the two given equations, and their sum is

$$5y + 5z = 150$$

Dividing by 5, and $y + z = 30$

But, $y + 4z = 48$

By subtracting, $3z = 18$

Hence, $z = 6$, and $y = 30 - 6 = 24$, *Ans.*

(Page 147.)

10. Given $\begin{cases} 2x + 3y = 7 & (1) \\ 4x + 5y = 13 & (2) \end{cases}$

Multiply (1) by 2, and $4x + 6y = 14$

But, $4x + 5y = 13$

*Diff.* $y = 1.$

Now (1) becomes $2x + 3 = 7$, $2x = 4$, or, $x = 2$.

11. Given $5y + 3x = 93$ (1)

and $3y + 4x = 80$ (2)

By subtracting, $2y - x = 13$ (3)

Multiply (3) by 3, $6y - 3x = 39$ (4)

Add (1) to (4), and $11y = 132$, and $y = 12$

This value of $y$, subtracted in (2), and $36 + 4x = 80$; $9 + x = 20$; and $x = 11$.

12. Given $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 14 & (1) \\ \frac{1}{3}x + \frac{1}{2}y = 11 & (2) \end{cases}$

By adding, $\frac{5}{6}x + \frac{5}{6}y = 25$, or $\frac{x}{6} + \frac{y}{6} = 5$

$x + y = 30$

Double (1), and $x + \frac{2}{3}y = 28$

By subtracting, $\frac{y}{3} = 2$

Hence, $y = 6$, and $x = 24$, *Ans.*

13. The sum of the given equations is

$$\frac{3}{2}x + \frac{3}{2}y = 15$$

Whence, $x + y = 10$

But, $x + \frac{1}{2}y = 8$

Hence, $\frac{1}{2}y = 2$; $y = 4$; and $x = 6$, *Ans.*

14. The sum of the given equations is

$$\frac{x+y}{7} + 7(x+y) = 150. \quad \text{Place } x + y = s.$$

Then, $\frac{1}{7}s + 7s = 150$

Whence, $s + 49s = 150 \times 7$

$50s = 150 \times 7$

$s = 3 \times 7 = 21$

That is, $x + y = 21$.

From the 2d of the given equations, we have

$$49x + y = 357$$

But, $\quad x + y = 21$

Diff. $\quad 48x = 336$

$x = 7$; whence $y = 14$, *Ans.*

15. Given $\begin{cases} \dfrac{x-y}{5} + \dfrac{x+y}{19} = 4 & (1) \\ x - y = 10 & (2) \end{cases}$

Observe that (2) shows us that the first numerator in (1) is 10. That is, $x - y = 10$; and, when divided by 5, the quotient must be 2. Then 2 dropped from each side of (1), and

$$\frac{x+y}{19} = 2.$$

Whence, $\quad x + y = 38.$

But, $\quad x - y = 10.$

Sum, $\quad 2x = 48.$

$x = 24$, and $y = 34$, *Ans.*

16. Add the given equations, and divide by 2, and then $x = a + b$.

Subtracting and dividing by 2, and $y = a - b$.

17. Dividing the first equation by 2, and we shall have

$2x - y = 4d - a. \quad (1)$

Add $\quad x + y = 2d + a + 3c. \quad (2)$

Sum, $\quad 3x = 6d + 3c.$

Dividing by 3, and $\quad x = 2d + c, \quad (3)$

Subtracting (3) from (2), and $\quad y = a + 2c.$

18. Add the given equations, and divide by $2c$. Then,

$$x = \frac{a+b}{c}$$

Subtract the second from the first, and divide by $2m$, and

$$y = \frac{a-b}{m}.$$

19. By subtraction we obtain $(b+c)y = b - c$.

Whence, $$y = \frac{b-c}{b+c}.$$

This value of $y$, multiplied by $c$, and the product, substituted in the first equation, will give $ax + \dfrac{bc - c^2}{b+c} = b$.

Clearing of fractions, and we have

$$a(b+c)x + bc - c^2 = bc + b^2.$$

Whence, $$x = \frac{b^2 + c^2}{a(b+c)}.$$

20. Multiply the first of the given equations by $m^2$, and the second by $n$. Then, we shall have

$$mx + m^2ny = m^3 + m^2n;$$

and $$mx + \frac{n^4y}{m} = m^2n + n^3.$$

By subtraction, $\left(m^2n - \dfrac{n^4}{m}\right)y = m^3 - n^3.$

Multiply by $m$, and $(m^3n - n^4)\,y = (m^3 - n^3)m;$

or, $(m^3 - n^3)ny = (m^3 - n^3)m.$

Dividing each side by the common factor, $m^3 - n^3$, and we have $ny = m$; or, $y = \dfrac{m}{n}$

In place of $ny$ in the first equation, write its equal $m$, and then we shall have $\dfrac{x}{m} + m = m + n;$

or, $\dfrac{x}{m} = n$; or, $x = mn.$

## PROBLEMS CONTAINING TWO UNKNOWN QUANTITIES.

(**153**, page 150.)

5. Let $\frac{x}{y}$ represent the fraction.

Then, by the conditions, $\frac{x+4}{y}=\frac{1}{2}$, and $\frac{x}{y+7}=\frac{1}{5}$.

Whence, $2x+8=y$, (1) and $5x=y+7$, (2)

Subtracting (1) from (2), $3x=15$; and $x=5$.

But, $y=2x+8=18$; whence, $\frac{5}{18}$ is the fraction sought.

6. Let $x=$ A's money, and $y=$ B's. Then, if B give A \$15, A will have $x+15$, and B will have $y-15$.

Now, by the first condition, $x+15=5y-75$ (1).

Again, if A give B \$5, A will have $x-5$, and B will have $y+5$. Then, by the second condition, $x-5=y+5$ (2).

Subtract (2) from (1), and $20=4y-80$. Whence, $4y=100$, and $y=25$.

But, $x-5=y+5$. That is, $x-5=25+5$;

or, $x=35$, A's money.

NOTE. Although it is more natural and easy to use two symbols in solving problems like those at present under consideration, yet we can sometimes solve them by the use of a single letter. For example, take the last problem. Let $x=$ A's money; then, if B give A \$15, A will have $x+15$. This, divided by 5, will give B's money, less \$15 that he has just given to A.

Then, B had at first $\frac{x+15}{5}+15$ dollars.

Now, if A gives B \$5, B will have $\frac{x+15}{5}+20$, and A will have $x-5$.

But now they will have the same number of dollars;

$$\text{that is, } x-5=\frac{x+15}{5}+20;$$

$$5x-25=x+15+100;$$

or, $4x=140$, and $x=35$, A's money, as before.

7. Let $x =$ the numerator, and $y =$ the denominator of the fraction in question.

Then, $\frac{2x}{y+7} = \frac{2}{3}$ (1), by one condition;

and $\frac{x+2}{2y} = \frac{3}{5}$ (2), by the other.

From (1) $6x = 2y + 14$ (3)

From (2) $5x + 10 = 6y$ (4)

3 times (3) $18x = 6y + 42$ (5)

(4) from (5) $13x - 10 = 42$

$13x = 52$; whence $x = 4$.

This value of $x$, substituted in (3), gives $24 = 2y + 14$;

or, $12 = y + 7$, or $y = 5$.

Therefore, the fraction is $\frac{4}{5}$, *Ans.*

8. Let $x =$ A's money, and $y =$ B's.

Then, if A give B \$5, A will have $x - 5$, and B will have $y + 5$.

By the first condition, $2x - 10 = y + 5$ (1).

Again, if B give A \$5, A will have $x + 5$, and B will have $y - 5$.

And, by the second condition, we have

$x + 5 = 3y - 15$ (2)

Double (2), and $2x + 10 = 6y - 30$ (3)

Subtracting, (1) $20 = 5y - 35$;

or, $5y = 55$, and $y = 11$.

11 substituted for $y$ in (1), and $x = 13$.

Hence, A has \$13; and B \$11.

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6*

9. Let $x$ = the number of pounds of 9 cent sugar;
and $y$ = the number of pounds of 13 cent sugar.

$$\text{Then,} \quad x + y = 100 \qquad (1)$$
$$\text{and} \quad 9x + 13y = 12x + 12y \qquad (2)$$
$$y = 3x.$$

$3x$ written for $y$ in (1), produces $4x = 100$.
Whence $x = 25$; and $y = 75$, *Ans.*

10. Let $x$ = the value of the better horse;
and $y$ = the value of the poorer;

$$\left.\begin{array}{ll} y + 50 = 2x & (1) \\ x + 50 = 3y & (2) \end{array}\right\} \text{by conditions.}$$

Double (2) $\quad 2x + 100 = 6y \qquad (3)$
From (1) $\quad 2x - 50 = y \qquad (4)$

Subtract (4) from (3), and $150 = 5y$, or $y = 30$.
Substitute this value in (1), and $30 + 50 = 2x$;
whence, $\quad x = 40$.

11. Let $x$ = the daily wages of the men;
and $y$ = that of the boys.

$$\left.\begin{array}{ll} \text{Then,} & 4x + 8y = 40 \\ \text{And,} & 7x + 6y = 50 \end{array}\right\} \text{by conditions.} \qquad \begin{array}{l}(1)\\(2)\end{array}$$

$\frac{1}{4}$ of (1) $\quad x + 2y = 10 \qquad (3)$
3 times (3) $\quad 3x + 6y = 30 \qquad (4)$
(4) from (2) $\quad 4x = 20.$

Whence, $x = 5$; then, $x = 5$, taken from (3), and $2y = 5$,
or, $y = 2\frac{1}{2}$.

12. Let $x =$ the price of a yard of broadcloth;
and $y =$ the price of a yard of velvet.

$$\left.\begin{array}{rl} \text{Then,}\quad x + 3y = 25 & (1) \\ 4x + 5y = 65 & (2) \end{array}\right\} \text{by conditions.}$$

Multiply (1), by 4, $4x + 12y = 100$ (3)

Subtract (2) from (3), $7y = 35$

or, $y = 5$

The value of $3y$, subtracted from each member of (1), gives

$$x = 10$$

13. Let $x =$ the first number, and $y =$ the second.

Then, by the conditions, $\frac{x}{2} + \frac{y}{3} = 9$ (1)

and $\frac{x}{4} + \frac{y}{5} = 5$ (2)

From the double of (2) subtract (1), and we obtain,

$$\frac{2y}{5} - \frac{y}{3} = 1;\ \text{whence, } y = 15.$$

Take $\frac{1}{3}$ of $y$ from each member of (1), and

$$\frac{x}{2} = 4;\ \text{whence, } x = 8.$$

14. Let $x =$ the age of the elder son, and $y =$ the age of the younger. Then, by the conditions, we have

$$x + y + 18 = 2x \quad (1)$$
$$x - y - 6 = y \quad (2)$$

By addition, $2x + 12 = 2x + y$, or $y = 12$.

This value of $y$ substituted in (1), and $x = 30$.

15. Let $x =$ A's money; and $y =$ B's.

Then, $x + 100 = y - 100$ (1);

and, $2(x - 100) = y + 100$ (2).

Subtract (1) from (2), $x - 300 = 200$, or $x = 500 =$ A's.

This value substituted in (1), and $y = 700 =$ B's.

16. Let $x =$ the first number; and $y =$ the second.

$$\text{Then,} \qquad \frac{3x}{5} + \frac{2y}{7} = 12 \qquad (1)$$

$$\tfrac{2}{3}\left(\frac{x}{2} + 3y\right) = 26 \qquad (2)$$

Five times (1), and $3x + \dfrac{10y}{7} = 60$ (3)

From (2) $x + 6y = 78$ (4)

Multiply (4) by 3, and $3x + 18y = 234$ (5)

(3) from (5), and $18y - \dfrac{10y}{7} = 174$

$$126y - 10y = 174 \times 7$$

Or, $116y = 174 \times 7$

$$58y = 87 \times 7 = 609$$

$$y = \frac{609}{58} = 10\tfrac{1}{2}, \textit{Ans.}$$

$6y = 63$. Hence, 63 substituted in (4) gives $x = 15$.

17. Let $x =$ A's money; and $y =$ B's. Then,

$$\left.\begin{array}{ll} \dfrac{x-y}{3} = y. & (1) \\ x + 2 = 5(y-2). & (2) \end{array}\right\} \text{by the conditions.}$$

From (1) $x = 4y$; and this value of $x$ placed in (2), that equation becomes $4y + 2 = 5y - 10$.

Hence, $y = 12$; and $x = 4y = 48$, *Ans.*

18. Let $x =$ the number of eggs at $\frac{1}{2}$ cent each;

and $y =$ " " $\frac{2}{3}$ of a cent each.

Then, the whole cost, $\dfrac{x}{2} + \dfrac{2}{3}y = 65$. (1)

Now, $x$ eggs were sold at 1 cent each, and $y$ eggs were sold at $\frac{2}{3} + \frac{1}{2}$, or $\frac{7}{6}$ of a cent each. Hence, the sale was,

$$x + \frac{7y}{6} = 120. \qquad (2)$$

Double (1), and $x + \frac{4y}{3} = 130.$ (3)

Subtracting (2) from (3), and $\frac{y}{6} = 10$; or, $y = 60.$

This value of $y$ placed in (2), will give $x = 50.$

20. Is solved in the book, by placing $a = 28$, and $b = 6.$

21. Let $x =$ the greater number; and $y =$ the less.

Then, $x + y = 100$; (1)

and $2x - 3y = 150.$ (2)

Double (1), and subtract (2), and we obtain

$5y = 50$; or, $y = 10$; whence, $x = 90.$

22. Let $x =$ the daily wages of the husband;

and $y =$ " " " wife.

Then, $mx + my = 2a$ (1) by the 1st condition;

and $mx - my = 2c$ (2) " 2d "

By addition, $2mx = 2a + 2c.$

$$x = \frac{a + c}{m}, \text{ man's wages.}$$

Subtract (2) from (1), and $2my = 2a - 2c.$

$$y = \frac{a - c}{m}, \text{ wife's wages.}$$

23. Let $x =$ one number;

and $y =$ the other.

Then, $2x - y = 3b$ (1), by the 1st condition;

$2y - x = 3a$ (2), " 2d "

By addition, $x + y = 3a + 3b$ (3)

Sum of (1) and (3), $3x = 3a + 6b$, or $x = a + 2b.$

Sum of (2) and (3), $3y = 6a + 3b$, or $y = 2a + b.$

24 Let $x$ and $y$ represent the numbers.

Then, $x + y = a;$ (1)

and, $x : y :: m : n;$

or, $nx = my.$ (2)

Multiply (1) by $n$, and $nx + ny = na;$

whence, $my + ny = na;$

$$y = \frac{na}{m+n}; \text{ and } x = \frac{ma}{m+n}, \textit{ Ans.}$$

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## SIMPLE EQUATIONS,

### CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

(**154**, page 155.)

$$\text{ven} \left\{ \begin{array}{ll} 3x + 9y + 8z = 41 & (1) \\ 5x + 4y - 2z = 20 & (2) \\ 11x + 7y - 6z = 37 & (3) \end{array} \right\} \text{to find } x, y, z.$$

Here we observe that the coefficient of $z$, in the 2d equation, can be made numerically equal to the coefficients of $z$ in the other two equations, by multiplying it by 2 and 4.

That is, the 2d, multiplied by 3, gives

$$15x + 12y - 6z = 60$$

Subtracting (3), $4x + 5y = 23$ (5)

Equation (2) multiplied by 4, gives

$$20x + 16y - 8z = 80 \quad (6)$$

Add (1), $23x + 25y = 121$ (7)

5 times (5), $20x + 25y = 115$ (8)

Diff. $3x = 6$; whence, $x = 2.$

Substitute this value of $x$, in equation (5), and we have,

$$8 + 5y = 23; \text{ or, } 5y = 15; \text{ and } y = 3.$$

Substituting the values of $x$ and $y$ in (2), and we obtain,

$$10 + 12 - 2z = 20; \text{ and } z = 1.$$

2. The first equation doubled, and we have

$$6x + 10y + 2z = 52$$

Subtract, second; $7y = 21$, and $y = 3$.

5. Add all three, and divide the sum by 3, and $x = 12$;

Whence, $y = 8$, and $z = 6$.

6. Place $a = 6$; then $2a = 12$, and $4a = 24$.

| | | |
|---|---|---|
| Assume | $x + y + z = s$ | (A) |
| Add | $x - y - z = a$ | |
| Sum, | $2x = a + s$ | (1) |
| To | $x + y + z = s$ | |
| Add | $3y - x - z = 2a$ | |
| Sum, | $4y = 2a + s$ | (2) |
| To | $x + y + z = s$ | |
| Add | $7z - y - x = 4a$ | |
| | $8z = 4a + s$ | (3) |
| Double (2), and | $8y = 4a + 2s$ | (4) |
| Multiply (1) by 4, and | $8x = 4a + 4s$ | (5) |
| Sum of (3), (4), (5), | $8s = 12a + 7s$; | |

Subtracting $7s$ from both members, and $s = 12a = 72$.

Substituting the values of $a$ and $s$ in (1), and we have,

$$2x = 78, \text{ or } x = 39.$$

Substituting in (2), and $4y = 12 + 72$, or $y = 3 + 18 = 21$

" (3), and $8z = 24 + 72$, or $z = 3 + 9 = 12$

The common reduction of these equations is very easy; but we give the foregoing in the hope of cultivating a higher standard of algebraic taste.

By subtracting the first of the given equations from the second, we obtain, $4y - 2x = 6$ (A)

Multiply the first equation by 7, and to the product add the third. That is,

$$\begin{array}{ll} \text{To} & 7x - 7y - 7z = 42 \\ \text{Add} & \underline{7z - \ y - \ x = 24} \\ \text{Sum,} & 6x - 8y = 66 \\ \text{or,} & 3x - 4y = 33 \\ \text{Add equation (A),} & \underline{4y - 2x = \ 6} \\ \text{Sum,} & x = 39 \end{array}$$

7. $\begin{cases} x + \frac{1}{2}y = a = 100 & (1) \\ y + \frac{1}{3}z = a & (2) \\ z + \frac{1}{4}x = a & (3) \end{cases}$

Multiply (2) by 3, and from the product subtract (3).

$$\begin{array}{ll} \text{Thus,} & 3y + \ z = 3a \\ & \underline{\frac{1}{4}x + \ z = \ a} \\ & 3y - \frac{1}{4}x = 2a \quad (4) \end{array}$$

We have now two equations, (1) and (4), involving two unknown symbols only. Clear of fractions in both (1) and (4);

$$\begin{array}{lll} \text{and we have,} & 2x + \ y = \ 2a & (5) \\ \text{and} & -x + 12y = \ 8a & (6) \\ \text{Double (6),} & -2x + 24y = 16a & \\ \text{Add (5),} & 25y = 18a = 18 \times 100 & \\ & y = 18 \times 4 = 72. & \end{array}$$

The value of $\frac{1}{2}y$, substituted in (1), and $x + 36 = 100$,

or, $x = 64$.

8. $\begin{cases} x + y = 52 & (1) \\ y + z = 82 & (2) \\ z + w = 68 & (3) \\ w + u = 30 & (4) \\ u + x = 32 & (5) \end{cases}$

Sum, $2x + 2y + 2z + 2w + 2u = 264$

$\frac{1}{2}$ sum, $(x+y)+z+(w+u)=132$.

That is, $52+z+30=132$; whence, $z=50$.

This value of $z$, taken for each member of (2), and $y=32$.

This value of $y$, taken for each member of (1), and $x=20$.

This value of $x$, taken from each member of (5), and $u=12$.

This value of $u$, taken from each member of (4), and $w=18$.

9. Reduce the equations.
$$\left\{\begin{array}{rr}\frac{x}{3}+3y=23 & (1)\\ x+\frac{1}{4}z=\ \ 8 & (2)\\ y+3z=31 & (3)\\ x+y+z+2w=39 & (4)\end{array}\right. \quad Ans. \left\{\begin{array}{l}x=6,\\ y=7,\\ z=8,\\ w=9.\end{array}\right.$$

$$\begin{array}{rr} x+9y=69 & (5)\\ 4x+\ \ z=32 & (6)\\ y+3z=31 & (7)\end{array}$$

Multiply (6) by 3, and from the product subtract (7), and we shall have

$$12x-\ \ y=65$$

Multiply by 9, and $108x-9y=65\times 9$

Add (5) $\qquad 109x=65\times 9+69=654$

Whence, $\qquad x=\frac{654}{109}=6$

$\frac{x}{3}=2$, subtracted from (1), and $3y=21$, or $y=7$.

The value of $x$, put in (2), and $z=8$. $x+y+z$, taken from (4), and $2w=18$, and $w=9$.

10. Given $\left\{\begin{array}{r}4x+2y-3z=\ \ 4\\ 3x-5y+2z=22\\ x+\ \ y+\ \ z=12\end{array}\right\}$ to find $x$, $y$, and $z$. (1) (2) (3)

Double (3), and from the product subtract (2)

Then, $\qquad -x+\ \ 7y=2 \qquad (4)$

Multiply (3) by 3, and to the product add (1).

Then, $\qquad 7x + 5y = 40 \qquad (5)$

7 times (4) gives $\quad -7x + 49y = 14 \qquad (6)$

Add (5) and (6), $\qquad 54y = 54$; or, $y = 1$.

Substitute this value of $y$, in (5), and we have

$$7x + 5 = 40; \text{ whence } 7x = 35; \text{ and } x = 5.$$

11. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 22 & (1) \\ \frac{1}{4}x + \ \ y + \frac{1}{2}z = 33 & (2) \\ \ \ x + \frac{1}{4}y - \frac{1}{8}z = 19 & (3) \end{cases}$

Clear the coefficients of $z$ of fractions. Then,

$$2x + \tfrac{4}{3}y + z = 88 \qquad (4)$$

$$\tfrac{1}{2}x + 2y + z = 66 \qquad (5)$$

$$8x + 2y - z = 152 \qquad (6)$$

Add (6) to (4), and to (5), and there will result,

$$10x + 3\tfrac{1}{3}y = 240 \qquad (7)$$

and $\quad 8\frac{1}{2}x + 4y = 218 \qquad (8)$

Multiply (7) by 3, and $\qquad 30x + 10y = 720$;

or, $\qquad 3x + y = 72 \qquad (9)$

Multiply (9) by 4, and $\qquad 12x + 4y = 288$

Subtract (8), and $\qquad 3\frac{1}{2}x = 70$

Whence, $\qquad 7x = 70 \times 2$; and $x = 20$.

12. $\qquad x + y = a \qquad (1)$

$\qquad x + z = b \qquad (2)$

and, $\qquad y + z = c. \qquad (3)$

The sum of (1) and (2) is $\qquad 2x + y + z = a - b$;

but, (3) $\qquad y + z = c$

Whence, by subtraction and division, $\qquad x = \frac{1}{2}(a + b - c)$.

**18.** From the 3d of the given equations subtract the 2d, and we have

$$(ac-c^2)x+y+(ac-a^2)z = a^2-ac+c^2; \quad (1)$$

subtract $\quad cx+y+az = a+ac+c. \quad (2)$

Diff. $(ac-c^2-c)x+(ac-a^2-a)z = a^2-a-ac+c^2-c-ac; \quad (3)$

or, $(a-c-1)cx+(c-a-1)az = (a-c-1)a+(c-a-1)c; \quad (4)$

Now, we may assume that

$$(a-c-1)cx = (a-c-1)a; \quad (5)$$

then will $\quad (c-a-1)az = (c-a-1)c. \quad (6)$

$$\text{From (5)}, \quad cx = a; \text{ or, } x = \frac{a}{c}.$$

$$\text{From (6)}, \quad az = c; \text{ or, } z = \frac{c}{a}.$$

These values of $cx$ and $az$, substituted in (3), will give us

$$a+y+c = a+ac+c;$$

Or, $\quad y = ac.$

## PROBLEMS PRODUCING EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

( **156**, page 157. )

1. Let A, B, C represent the ages sought.

$$A+B+C = 90 \text{ years} \quad (1)$$
$$A+B \quad = 50 \quad \text{"} \quad (2)$$
$$B+C = 70 \quad \text{"} \quad (3)$$

Subtracting (2) from (1), and $C = 40$.

Subtracting (3) from (1), and $A = 20$.

The sum of A and C, 60, taken from (1), and $B = 30$, *Ans.*

3. Let $x =$ the number of sheep in the first pasture;

$y =$ " " second "

$z =$ " " third "

Then, by the conditions, we obtain the following equations:

$$x + \tfrac{1}{2}y + \tfrac{1}{2}z = 70 = a \quad (1)$$

$$y + \tfrac{1}{3}x + \tfrac{1}{3}z = 60 = b \quad (2)$$

$$z + \tfrac{1}{5}x + \tfrac{1}{5}y = 58 = c \quad (3)$$

From (1) $\quad x + y + z = 2a - x \quad (4)$

From (2) $\quad x + y + z = 3b - 2y \quad (5)$

From (6) $\quad x + y + z = 5c - 4z \quad (6)$

Now, let $x + y + z = S$; and,

(4) becomes, $\quad S = 2a - x \quad (7)$

(5) becomes, $\quad S = 3b - 2y \quad (8)$

(6) becomes, $\quad S = 5c - 4z \quad (9)$

Now, multiply (7) by 4, and (8) by 2; and we shall have,

$$4S = 8a - 4x \quad (10)$$

$$2S = 6b - 4y \quad (11)$$

and (9), $\quad S = 5c - 4z$

By addition, $\quad 7S = 8a + 6b + 5c - 4S;$

or, $\quad 11S = 8a + 6b + 5c,$

$$S = \frac{8a + 6b + 5c,}{11} = 110.$$

This value of S, put in (7), and $110 = 140 - x$;

or, $x = 30$.

From (8) $2y = 3b - 110 = 180 - 110 = 70$;

whence, $y = 35$.

From (9) $4z = 5c - 110 = 290 - 110 = 180$;

whence, $z = 45$.

4. Here we have, at once,

$$A + B = 900 \quad (1)$$
$$A + C = 800 \quad (2)$$
$$B + C = 700 \quad (3)$$

By addition, $2(A + B + C) = 2400$

$$A + B + C = 1200 \quad (4)$$

From (4) subtract (1), and $C = 300$

From (4) subtract (2), and $B = 400$

From (4) subtract (3), and $A = 500$

$$\left.\begin{array}{l} C = 300 \\ B = 400 \\ A = 500 \end{array}\right\} Ans.$$

5. Let $x$, $y$, and $z$ represent the three numbers;

then, $x + y + z = 59 \quad (1)$

$$x - y = 10 \quad (2)$$
$$x - z = 18 \quad (3)$$

By addition, $3x = 87$;

whence, $x = 29$.

This value of $x$, substituted for $x$ in (2) and (3), will show that $y = 19$, and $z = 11$, *Ans.*

6. Let $x =$ the number of tens;

and $y =$ " units.

Then the number, expressed in units, must be

$$10x + y.$$

By the conditions we obtain,

$$10x + y = 4x + 4y; \quad (1)$$

and $10x + y + 27 = 10y + x. \quad (2)$

From (1), $6x = 3y$; or, $y = 2x$.

From (2), $9x + 27 = 9y$.

Divide by 9, and $x + 3 = y = 2x$;

whence, $x = 3$, and $y = 6$;

and $36 =$ the number sought.

7. Let $x =$ the number of hundreds;

$y =$ " tens;

and $z =$ " units.

Then the number, expressed in units, must be

$$100x + 10y + z.$$

By the given conditions, we have

$$x + y + z = 9; \quad (1)$$

$$z = 2x; \quad (2)$$

$$100x + 10y + z + 198 = 100z + 10y + x. \quad (3)$$

Reduce (3), and $99x + 198 = 99z$;

or, $x + 2 = z = 2x$; see (2)

whence, $x = 2$; and $z = 4$;

take values of $x$ and $z$ from (1), and $y = 3$.

Hence, the number must be 234.

8. Let $x$, $y$ and $z$ represent the parts. Then,

$$x + y + z = 90, \quad (1)$$

$$2x + 40 = 3y + 20, \quad (2)$$

$$4z + 10 = 3y + 20. \quad (3)$$

From (2), $y = \frac{2x + 20}{3}$; (4)

(3) from (2), and $2x - 4z + 30 = 0$.

Whence, $z = \frac{x + 15}{2}$. (5)

Now, (4) and (5) substituted in (1), and

$$x + \frac{2x + 20}{3} + \frac{x + 15}{2} = 90; \quad (6)$$

$$6x + 4x + 40 + 3x + 45 = 540,$$

$$13x = 455,$$

$$x = 35.$$

But, $y = \frac{2x + 20}{3} = 30$;

and $z = \frac{35 + 15}{2} = 25$.

9. Let $x$, $y$ and $z$ represent the three numbers in question. Then, by the conditions,

$$x+\frac{y+z}{3}=25=a; \quad (1)$$

$$y+\frac{x+z}{4}=25=a; \quad (2)$$

$$z+\frac{x+y}{5}=25=a; \quad (3)$$

From (1), $x+y+z=3a-2x$; (4)

From (2), $x+y+z=4a-3y$; (5)

From (3), $x+y+z=5a-4z$; (6)

Now, assume $x+y+z=s$;

and (4), (5) and (6) become

$$s=3a-2x; \quad (7)$$

$$s=4a-3y; \quad (8)$$

$$s=5a-4z. \quad (9)$$

Now multiply (7) by 6, (8) by 4, (9) by 3, and we shall have

$$6s=18a-12x;$$

$$4s=16a-12y;$$

$$3s=15a-12z.$$

By addition, $13s=49a-12s$.

By transposition, $25s=49a=49\times 25$.

Whence, $s=49$.

This value of $s$, placed in (7), (8) and (9), will give the values of $x$, $y$ and $z$.

Then (7) becomes $49=75-2x$;

or, $x=13$;

and $y=17$;

$z=19$.

10. Let $x$, $y$, and $z$ represent the three numbers.

Then, by the conditions,

$$x + \tfrac{1}{2}y = 14 \quad (1)$$
$$y + \tfrac{1}{3}z = 18 \quad (2)$$
$$z + \tfrac{1}{4}x = 20 \quad (3)$$

From the double of (1), subtract (2), and we obtain,

$$2x - \tfrac{1}{3}z = 10$$

Multiply by 3, and

$$6x - z = 30 \quad (4)$$

Add (3), $\quad 6x + \tfrac{1}{4}x = 50$

$$25x = 50 \times 4$$
$$x = 8;$$

whence, $\quad \tfrac{1}{2}y = 6;\ y = 12;$ and $z = 18.$

11. Let $x =$ the price of a pair of shoes;

$y =$ " " coarse boots;

and $z =$ " " fine boots.

$$3x + 2y = 12 \quad (1)$$
$$y + 2z = 12 \quad (2)$$
$$4x + 3z = 21 \times 50 \quad (3)$$

Double (2), and subtract (1); then,

$$4z - 3x = 12 \quad (4)$$

Multiply (4) by 4, $\quad 16z - 12x = 48$

3 times (3), $\quad 9z + 12x = 63 \times 50$

By addition $\quad 25z = 111 \times 50$

$$z = 4\tfrac{1}{2} \text{ dollars; or, } 4.50.$$

The value of $2z$, taken from (2), and $y = 3$;

hence, $\quad 3x + 6 = 12;\ x + 2 = 4;\ x = 2.$

12. Let $x$, $y$, $u$, and $z$ represent the four sums of money. Then, by the conditions,

$$x + \frac{y+u+z}{3} = 30 = a \qquad (1)$$

$$y + \frac{x+u+z}{3} = 32 = a+2 \qquad (2)$$

$$u + \frac{x+y+z}{3} = 34 = a+4 \qquad (3)$$

$$z + \frac{x+y+u}{3} = 36 = a+6 \qquad (4)$$

From (1), $x+y+u+z = 3a - 2x$

or, $s = 3a - 2x \qquad (5)$

From (2), $s = 3a + 6 - 2y \qquad (6)$

From (3), $s = 3a + 12 - 2u \qquad (7)$

From (4), $s = 3a + 18 - 2z \qquad (8)$

By addition, $4s = 12a + 36 - 2s$

By transposition, $6s = 12a + 36$

By division, $s = 2a + 6.$

This value of $s$, written in (5), and we obtain,

$$2a + 6 = 3a - 2x;$$

whence, $2x = a - 6$; or, $x = \frac{1}{2}a - 3 = 12.$

Substituting in the same manner in (6), (7), and (8), we obtain, $y$, $u$, and $z$.

14. Let $\frac{x}{y}$ = the first fraction. Then, by the conditions given, $\frac{2x}{y}$ = the second fraction, and $\frac{4x}{y}$ = the third.

Then, $\frac{7x}{y} = 2$, and $x = \frac{2}{7}$, the first fraction; consequently, $\frac{2}{7}$, $\frac{4}{7}$, and $\frac{8}{7}$ are the fractions sought.

15. Let $x$, $y$, and $z$ represent the three numbers.

Then, by the conditions,

$$x + \frac{y+z}{3} = 23 \quad (1)$$

$$y + \frac{x+z}{2} = 30 \quad (2)$$

$$z + 2(x+y) = 72 \quad (3)$$

From (1), $3x + y + z = 69$ (4)

From (2), $x + 2y + z = 60$ (5)

By subtraction, $2x - y = 9$ (6)

Subtracting (5) from (3), and $x = 12$.

Whence, (6) becomes, $24 - y = 9$; or, $y = 15$,

and (3) becomes, $z + 54 = 72$; or, $z = 18$.

16. Let $x$=C's age. Then, $3x$=B's age, and $6x$=A's age.
Whence, $10x = 140$; $x = 14$, C's age, *Ans.*

17. Let $x$ = the daily wages of the husband;

$y$ = " " " wife;

and $z$ = " " " son.

Then, by conditions,

$$10x + 4y + 3z = 11.50 \quad (1)$$

$$9x + 8y + 6z = 12.00 \quad (2)$$

$$7x + 6y + 4z = 9.00 \quad (3)$$

From the double of (1), subtract (2), and $11x = 11.00$; or, $x = 100$ cents.

This value of $x$, being substituted in (2) and (3), and reduced, we obtain

$$8y + 6z = 3.00 \quad (4)$$

and, $6y + 4z = 2.00$ (5)

(5) divided by 2, $3y + 2z = 1.00$ (6)

(5) from (4), $2y + 2z = 1.00$ (7)

(7) from (6), and $y = 0$

Whence, $z = \frac{1}{2}$, or 50 cents.

## INTERPRETATION OF NEGATIVE RESULTS.

( **157**, page 161. )

1. Let $x =$ the daily wages of the husband;

$y =$ " " wife;

and $z =$ " " son.

Then, by the conditions,

$$10x + 8y + 6z = 10.30 \quad (1)$$
$$12x + 10y + 4z = 13.20 \quad (2)$$
$$15x + 10y + 12z = 13.85 \quad (3)$$

Subtract (2) from (3), and $3x + 8z = .65$ (4)

Divide (1) and (2) each by 2, then,

$$5x + 4y + 3z = 5.15 \quad (5)$$
$$6x + 5y + 2z = 6.60 \quad (6)$$

Multiply (5) by 5, and (6) by 4, and we have,

$$25x + 20y + 15z = 25.75$$
$$24x + 20y + 8z = 26.40$$

By subtraction, $x + 7z = -.65$ (7)

or, $8x + 21z = -1.95$

Subtract (4), $13z = -2.60$

and $z = -.20.$

Hence, the son was an expense for board, because the result is *minus wages;* that is, he received the opposite of wages.

This value of $z$, put in (7), and $x - 1.40 = .65$;

or, $x = .75.$

To obtain $y$, we must substitute the values of $x$ and $z$ in equation (1) or (5).

2. Let $x =$ B's money;

and $3x =$ A's "

After each gained, B had $x + 150$, and A had $3x + 400$.

Now, by the second condition, we have

$$3x + 400 = 2x + 300$$

$$x + 100 = 0; \text{ or, } x = -100.$$

That is, B had no money, but was actually in debt \$100; and A was in debt \$300.

3. Let $x =$ the number.

Then, by the condition, we must have

$$\frac{x}{4} - \frac{x}{3} = 12;$$

$$3x - 4x = 144; \text{ or, } -x = 144; \text{ or, } x = -144.$$

By applying our judgment to the question, we observe that the fourth part of any number *cannot* exceed its third part. *Hence the negative result.* It is the third part that exceeds the fourth part, and when so taken, the number is $+ 144$.

4. Let $x$ be the number of years that must elapse, and then his age will be $30 + x$;

and her age, $15 + x$.

Now, by the condition, we must have

$$30 + x = 45 + 3x; \text{ or, } 15 + 2x = 0; \text{ whence, } x = -7\tfrac{1}{2}.$$

Here the minus sign indicates that the years must be counted backwards; whence, the husband's age was $22\frac{1}{2}$ years, and the wife's age, $7\frac{1}{2}$ years. In this case, $7\frac{1}{2}$ multiplied by 3, will give $22\frac{1}{2}$, as required.

5. Let $\frac{x}{y} =$ the fraction.

Then, by conditions, $$\frac{x+1}{y} = \frac{3}{5};$$

and, $$\frac{x}{y+1} = \frac{5}{7}.$$

Thence, $$5x + 5 = 3y. \qquad (1)$$

$$7x = 5y + 5. \qquad (2)$$

Sum of (1) and (2), $$12x = 8y.$$

$$x = \frac{2y}{3}. \qquad (3)$$

This value of $x$, substituted in (1), will give

$$\frac{10y}{3} + 5 = 3y;$$

$$10y + 15 = 9y;$$

whence, $$y = -15.$$

This value of $y$, substituted in (3), gives

$$x = -10;$$

and the fraction is $\frac{-10}{-15}$. These are not numbers; but if we reduce the fraction algebraically, by dividing its terms by $-5$, we shall then have $\frac{2}{3}$ as an equivalent value. Hence, we can take the result only in its algebraic sense.

If, to the numerator of $\frac{-10}{-15}$, considered algebraically, we add 1, we shall then have $\frac{-9}{-15}$, or $\frac{3}{5}$; and if we add 1 to the denominator, we shall have $\frac{-10}{-14}$, or $\frac{5}{7}$.

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## INVOLUTION;

### OR, FORMATION OF POWERS.

We will take the first example, and perform it by what we shall call the primary method of operation, and then again by the more elegant and summary method developed and explained by J. H. French, called French's method.

(**179**, page 179.)

1. Required the 4th power of $2a + 3x$.

To put this into a simple literal binomial, we place $P = 2a$, and $Q = 3x$. We are now required to find the value of $(P + Q)^4$, in terms of $a$ and $x$.

$$(P + Q)^4 = P^4 + 4P^3Q + 6P^2Q^2 + 4PQ^3 + Q^4.$$

| | | | |
|---|---|---|---|
| But $P^4 = 16a^4$ | | | $P^4 = 16a^4$ |
| $4P^3 = 32a^3$ | $Q = 3x$ | | $4P^3Q = 96a^3x$ |
| $6P^2 = 24a^2$ | $Q^2 = 9x^2$ | | $6P^2Q^2 = 216a^2x^2$ |
| $4P = 8a$ | $Q^3 = 27x^3$ | | $4PQ^3 = 216ax^3$ |
| | $Q^4 = 81x^4$ | | $Q^4 = 81x^4$. |

Whence $(2a + 3x)^4 = 16a^4 + 96a^3x + 216a^2x^2 + 216ax^3 + 81x^4$, *Ans.*

When the powers are high, and the coefficients large or fractional, this method becomes very tedious; therefore, we abandon it, and take French's method.

The general formula is $(ax \pm by)^n$. The first coefficient we denote by $C_1$, the second by $C_2$, the third by $C_3$, and so on.

*General Formula for Coefficients.*

$$C_1 = a^n$$
$$C_2 = \frac{C_1 nb}{a}$$
$$C_3 = \frac{C_2(n-1)b}{2a}$$
$$C_4 = \frac{C_3(n-2)b}{3a}$$
$$= \&c.$$

In the given example, $a = 2$, $b = 3$, $x = a$, $y = x$, $n = 4$.

Then, $C_1 = a^n \qquad\qquad\qquad\qquad = 16$

$$C_2 = \frac{C_1 nb}{a} = \frac{16 \times 4 \times 3}{2} = 96$$

$$C_3 = \frac{C_2(n-1)b}{2a} = \frac{96 \times 3 \times 3}{2 \times 2} = 216$$

$$C_4 = \frac{C_3(n-2)b}{3a} = \frac{216 \times 2 \times 3}{3 \times 2} = 216$$

$$C_5 = \frac{C_4(n-3)b}{4a} = \frac{216 \times 1 \times 3}{4 \times 2} = 81$$

Here the operation must stop, because the factor in the next term $(n-4=0)$, is zero, making the whole zero, and, in fact, closing the operation as it ought to be.

Whence, $16a^4 + 96a^3x + 216a^2x^2 + 216ax^3 + 81x^4$ must be the required power.

2. Expand $(2a-5b)^3$. The general formula is $(ax \pm by)^n$. Here $a$ in the formula must equal 2, $x = a$, $b = 5$, $y = b$, and $n = 3$.

Then, $C_1 = a^n \qquad = 2^3 \qquad = 8$

$$C_2 = \frac{C_1 nb}{a} = \frac{8 \times 3 \times 5}{2} = 60$$

$$C_3 = \frac{C_2(n-1)b}{2a} = \frac{60 \times 2 \times 5}{2 \times 2} = 150$$

$$C_4 = \frac{C_3(n-2)b}{3a} = \frac{150 \times 1 \times 5}{3 \times 2} = 125$$

Whence, $8a^3 - 60a^2b + 150ab^2 - 125b^3$.

The second, and every alternate sign, must be minus, because $b$ in the general formula is minus, as applied to this example.

3. What is the cube of $7x + 2ay$. Here $a = 7$, $x = x$, $b = 2$, $y = ay$, and $n = 3$.

$$\begin{aligned}
C_1 &= a^n &&= 7^3 &&= 343\\
C_2 &= \frac{C_1 nb}{a} &&= \frac{343 \times 3 \times 2}{7} &&= 294\\
C_3 &= \frac{C_2(n-1)b}{2a} &&= \frac{294 \times 2 \times 2}{2 \times 7} &&= 84\\
C_4 &= \frac{C_3(n-2)b}{3a} &&= \frac{84 \times 1 \times 2}{3 \times 7} &&= 8
\end{aligned}$$

Whence. $343x^3 + 294x^2ay + 84xa^2y^2 + 8a^3y^3$ is the power required.

4. Expand $(5a - 2c)^5$. Here $a = 5$, $b = 2$, $x = a$, $y = c$, and $n = 5$.

$$\begin{aligned}
C_1 &= a^n &&= 5^5 &&= 3125\\
C_2 &= \frac{C_1 nb}{a} &&= \frac{3125 \times 5 \times 2}{5} &&= 6250\\
C_3 &= \frac{C_2(n-1)b}{2a} &&= \frac{6250 \times 4 \times 2}{2 \times 5} &&= 5000\\
C_4 &= \frac{C_3(n-2)b}{3a} &&= \frac{5000 \times 3 \times 2}{3 \times 5} &&= 2000\\
C_5 &= \frac{C_4(n-3)b}{4a} &&= \frac{2000 \times 2 \times 2}{4 \times 5} &&= 400\\
C_6 &= \frac{C_5(n-4)b}{5a} &&= \frac{400 \times 1 \times 2}{5 \times 5} &&= 32
\end{aligned}$$

Whence, $3125a^5 - 6250a^4c + 5000a^3c^2 - 2000a^2c^3 + 400ac^4 - 32c^5$, *Ans.*

5. Expand $(x^2 + 3y^2)^5$. Here, $a = 1$, $x = x^2$, $b = 3$ $y = y^2$, and $n = 5$.

$$\begin{aligned}
C_1 &= a^n && &&= 1\\
C_2 &= \frac{C_1 nb}{a} &&= \frac{1 \times 5 \times 3}{1} &&= 15\\
C_3 &= \frac{C_2(n-1)b}{2a} &&= \frac{15 \times 4 \times 3}{2} &&= 90\\
C_4 &= \frac{C_3(n-2)b}{3a} &&= \frac{90 \times 3 \times 3}{3 \times 1} &&= 270
\end{aligned}$$

$$C_5 = \frac{C_4(n-3)b}{4a} = \frac{270 \times 2 \times 3}{4 \times 1} = 405$$

$$C_6 = \frac{C_5(n-4)b}{5a} = \frac{405 \times 1 \times 3}{5 \times 1} = 243$$

Whence, $x^{10} + 15x^8y^2 + 90x^6y^4 + 270x^4y^6 + 405x^2y^8 + 243y^{10}$, *Ans.*

6. Expand $(2a^2 + ax)^3$. Here, $a = 2$, $x = a^2$, $b = 1$, $y = ax$, and $n = 3$.

$$C_1 = a^n \quad = 2^3 \quad = 8$$

$$C_2 = \frac{C_1nb}{a} \quad = \frac{8 \times 3 \times 1}{2} = 12$$

$$C_3 = \frac{C_2(n-1)b}{2a} = \frac{12 \times 2 \times 1}{2 \times 2} = 6$$

$$C_4 = \frac{C_3(n-2)b}{3a} = \frac{6 \times 1 \times 1}{3 \times 2} = 1$$

Whence, $8a^6 + 12a^5x + 6a^4x^2 + a^3x^3$.

7. Expand $(x-1)^6$. This is the most simple form of the binomial, and can be at once written out by the original formula.

8. Expand $(3x - 5)^3$. Here, $a = 3$, $x = x$, $b = 5$, $y = 1$, and $n = 3$.

$$C_1 = a^3 \quad = 3^3 \quad = 27$$

$$C_2 = \frac{C_1nb}{a} \quad = \frac{27 \times 3 \times 5}{3} = 135$$

$$C_3 = \frac{C_2(n-1)b}{2a} = \frac{135 \times 2 \times 5}{2 \times 3} = 225$$

$$C_4 = \frac{C_3(n-2)b}{3a} = \frac{225 \times 1 \times 5}{3 \times 3} = 125$$

Whence, $27x^3 - 135x^2 + 225x - 125$

9. Expand $(4a^3b - 2c^2)^4$. Here $a = 4$, $x = a^3b$, $b = 2$, $y = c^2$, and $n = 4$.

$$\begin{aligned}
C_1 &= a^n &&= 4^4 &&= 256 \\
C_2 &= \frac{C_1 nb}{a} &&= \frac{256 \times 4 \times 2}{4} &&= 512 \\
C_3 &= \frac{C_2(n-1)b}{2a} &&= \frac{512 \times 3 \times 2}{2 \times 4} &&= 384 \\
C_4 &= \frac{C_3(n-2)b}{3a} &&= \frac{384 \times 2 \times 2}{3 \times 4} &&= 128 \\
C_5 &= \frac{C_4(n-3)b}{4a} &&= \frac{128 \times 1 \times 2}{4 \times 4} &&= 16
\end{aligned}$$

Whence, $256a^{12}b^4 - 512a^9b^3c^2 + 384a^6b^2c^4 - 128a^3bc^6 + 16c^8$, *Ans.*

10. Expand $(\frac{1}{2}a + \frac{3}{4}x)^5$. Here $a = \frac{1}{2}$, $x = a$, $b = \frac{3}{4}$, $y = x$, and $n = 5$.

$$\begin{aligned}
C_1 &= a^n &&= (\tfrac{1}{2})^5 &&= \tfrac{1}{32} \\
C_2 &= \frac{C_1 nb}{a} &&= \frac{\frac{1}{32} \times 5 \times \frac{3}{4}}{\frac{1}{2}} &&= \tfrac{15}{64} \\
C_3 &= \frac{C_2(n-1)b}{2a} &&= \tfrac{15}{64} \times 4 \times \tfrac{3}{4} &&= \tfrac{45}{64} \\
C_4 &= \frac{C_3(n-2)b}{3a} &&= \frac{\frac{45}{64} \times 3 \times \frac{3}{4}}{3 \times \frac{1}{2}} &&= \tfrac{135}{128} \\
C_5 &= \frac{C_4(n-3)b}{4a} &&= \frac{\frac{135}{128} \times 2 \times \frac{3}{4}}{2} &&= \tfrac{405}{512} \\
C_6 &= \frac{C_5(n-4)b}{5a} &&= \frac{\frac{405}{512} \times 1 \times \frac{3}{4}}{5 \times \frac{1}{2}} &&= \tfrac{243}{1024}
\end{aligned}$$

Whence, $\frac{1}{32}a^5 + \frac{15}{64}a^4x + \frac{45}{64}a^3x^2 + \frac{135}{128}a^2x^3 + \frac{405}{512}ax^4 + \frac{243}{1024}x^5$, *Ans.*

In the text-book, the 3d and 4th coefficients are not reduced to their lowest terms.

12. Expand $\left(x+\frac{1}{2x}\right)^7$. Here $a=1$, $x=x$, $b=\frac{1}{2}$, $y \times \frac{1}{x}$, and $n=7$.

$$\begin{aligned}
C_1 &= a^n && = 1^7 && = 1 \\
C_2 &= \frac{C_1 nb}{a} && = \frac{1 \times 7 \times \frac{1}{2}}{1} && = \tfrac{7}{2} \\
C_3 &= \frac{C_2(n-1)b}{2 \times a} && = \frac{\frac{7}{2} \times 6 \times \frac{1}{2}}{2} && = \tfrac{21}{4} \\
C_4 &= \frac{C_3(n-2)b}{3 \times a} && = \frac{\frac{21}{4} \times 5 \times \frac{1}{2}}{3} && = \tfrac{35}{8} \\
C_5 &= \frac{C_4(n-3)b}{4 \times a} && = \frac{\frac{35}{8} \times 4 \times \frac{1}{2}}{4} && = \tfrac{35}{16} \\
C_6 &= \frac{C_5(n-4)b}{5 \times a} && = \frac{\frac{35}{16} \times 3 \times \frac{1}{2}}{5} && = \tfrac{21}{32} \\
C_7 &= \frac{C_6(n-5)b}{6 \times a} && = \frac{\frac{21}{32} \times 2 \times \frac{1}{2}}{6} && = \tfrac{7}{64} \\
C_8 &= \frac{C_7(n-6)b}{7 \times a} && = \frac{\frac{7}{64} \times 1 \times \frac{1}{2}}{7} && = \tfrac{1}{128}
\end{aligned}$$

Whence, $x^7 + \frac{7}{2}x^5 + \frac{21}{4}x^3 + \frac{35}{8}x + \frac{35}{16}x + \frac{21}{32}x^3 + \frac{7}{64}x^5 + \frac{1}{128}x^7$, *Ans.*

**Note.** When the formules become familiar, we can drop a great portion of them, as in the following examples.

13. Expand $(1+\frac{5}{2}x)^5$. Here $a=1$, $b=\frac{5}{2}$, $y=x$, $n=5$.

$$\begin{aligned}
C_1 &= && 1 \\
C_2 &= \frac{1 \times 5 \times \frac{5}{2}}{1} && = \tfrac{25}{2} \\
C_3 &= \frac{\frac{25}{2} \times 4 \times \frac{5}{2}}{2} && = \tfrac{125}{2} \\
C_4 &= \frac{\frac{125}{2} \times 3 \times \frac{5}{2}}{3} && = \tfrac{625}{4} \\
C_5 &= \frac{\frac{625}{4} \times 2 \times \frac{5}{2}}{4} && = \tfrac{3125}{16} \\
C_6 &= \frac{\frac{3125}{16} \times 1 \times \frac{5}{2}}{5} && = \tfrac{3125}{32}
\end{aligned}$$

Whence, $1 + \frac{25}{2}x + \frac{125}{2}x^2 + \frac{625}{4}x^3 + \frac{3125}{16}x^4 + \frac{3125}{32}x^5$.

14. Expand $(\frac{3}{2} - \frac{5}{3}x)^4$. Here $a = \frac{3}{2}$, $x = 1$, $b = \frac{5}{3}$, $y = x$, and $n = 4$.

$$C_1 = (\tfrac{3}{2})^4 \qquad = \tfrac{81}{16}$$

$$C_2 = \frac{\frac{81}{16} \times 4 \times \frac{5}{3}}{\frac{3}{2}} = -\tfrac{45}{2}$$

$$C_3 = \frac{\frac{45}{2} \times 3 \times \frac{5}{3}}{3} = \tfrac{75}{2}$$

$$C_4 = \frac{\frac{75}{2} \times 2 \times \frac{5}{3}}{3 \times \frac{3}{2}} = -\tfrac{250}{9}$$

$$C_5 = \frac{\frac{250}{9} \times \frac{5}{3}}{6} = \tfrac{625}{81}$$

Whence, $\frac{81}{16} - \frac{45}{2}x + \frac{75}{2}x^2 - \frac{250}{9}x^3 + \frac{625}{81}x^4$, *Ans.*

15. Expand $(x^2 - 3y^2)^5$. Here $a = 1$, $x = x^2$, $b = -3$, $y = y^2$, and $n = 5$.

$$C_1 = a^5 \qquad = 1$$

$$C_2 = \frac{1 \times 5 \times -3}{1} = -15$$

$$C_3 = \frac{15 \times 4 \times -3}{2} = 90$$

$$C_4 = \frac{90 \times 3 \times -3}{3} = -270$$

$$C_5 = \frac{270 \times 2 \times -3}{4} = 405$$

$$C_6 = \frac{405 \times 1 \times -3}{5} = -243$$

Whence, $x^{10} - 15x^8y^2 + 90x^6y^4 - 270x^4y^6 + 405x^2y^8 - 243y^{10}$, *Ans.*

NOTE. When $n$ is $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, or any other fraction, the formula is the same; but the result is then finding a root, in place of a power. This will be fully illustrated in future editions of the University Algebra.

# EVOLUTION;

## OR, THE EXTRACTION OF ROOTS.

( 187, page 186. )

Because the square of $(a + b)$ is $a^2 + 2ab + b^2$ and the cube of $(a + b)$, is $a^3 + 3a^2b + 3ab^2 + b^3$, and so on for other powers, therefore, in literal powers, we generally know the form of the expression, and consequently we know what the root must be without applying the rules for extraction.

Thus, in the following example:

What is the square root of $1 - 4b + 4b^2 + 2y - 4by + y^2$?

The first term of the root must be 1, and the last term $y$; and $-4b$, divided by 2, the double of 1, will give $-2b$.

Hence, we may conclude that the root sought must be

$$1 - 2b + y.$$

By squaring this quantity we shall find the given expression.

This is finding the root by inspection, which is frequently done in literal expressions.

But suppose we required the square root of

$$1 - 5b + 3b^2 + 2y - 4by + y^2.$$

This expression has the same number of terms, the same letters, the same signs, and the same general appearance as the preceding expression. *But it is not a square*, and consequently we cannot find its square root.

Hence, in algebraic expressions, whenever a square root, a cube root, or any other root is demanded, *it is understood that the given expression has such a root.*

3. What is the square root of $4x^4 - 4x^3 + 13x^2 - 6x + 9$?

The square root of the first term is $2x^2$. The second term, $-4x^3$, divided by twice $2x^2$, or $4x^2$, and $-x$, is the second term of the root. The square root of the last term 9, is 3. Hence, $2x^2 - x + 3$, must be the root sought, if the given quantity is really a square.

A more formal solution is as follows:

$$
\begin{array}{lr}
 & 4x^4 - 4x^3 + 13x^2 - 6x + 9(2x^2 - x + 3 \\
 & 4x^4 \qquad\qquad\qquad\qquad\qquad \\
\hline
 & -4x^3 + 13x^2 \qquad\qquad\qquad \\
4x^2 - x & -4x^3 + x^2 \qquad\qquad\qquad \\
\hline
 & 12x^2 - 6x + 9 \\
4x^2 - 2x + 3 & 12x^2 - 6x + 9
\end{array}
$$

4. Extract the square root of

$$
\begin{array}{lr}
 & 4x^4 - 16x^3 + 24x^2 - 16x + 4(2x^2 - 4x + 2 \\
 & 4x^4 \qquad\qquad\qquad\qquad\qquad \\
\hline
 & -16x^3 + 24x^2 \qquad\qquad\qquad \\
4x^2 - 4x & -16x^3 + 16x^2 \qquad\qquad\qquad \\
\hline
 & 8x^2 - 16x + 4 \\
4x^2 - 8x + 2 & 8x^2 - 16x + 4
\end{array}
$$

5. Extract the square root of

$$
\begin{array}{lr}
 & 16x^4 + 24x^3 + 89x^2 + 60x + 100(4x^2 + 3x + 10 \\
 & 16x^4 \qquad\qquad\qquad\qquad\qquad \\
\hline
 & 24x^3 + 89x^2 \qquad\qquad\qquad \\
8x^2 + 3x & 24x^3 + 9x^2 \qquad\qquad\qquad \\
\hline
 & 80x^2 + 60x + 100 \\
8x^2 + 6x + 10 & 80x^2 + 60x + 100
\end{array}
$$

6. Extract the square root of

$$4x^4-16x^3+8x^2+16x+4(2x^2-4x-2$$

$$\underline{4x^4}$$

$$-16x^3+8x^2$$

$$4x^2-4x \qquad \underline{-16x^3+16x^2}$$

$$-8x^2+16x+4$$

$$4x^2-8x-2 \qquad -8x^2+16x+4$$

7. Extract the square root of

$$x^2+2xy+y^2+6xz+6yz+9z^2(x+y+3z$$

$$\underline{x^2}$$

$$2xy+y^2$$

$$2x+y \qquad \underline{2xy+y^2}$$

$$6xz+6yz+9z^2$$

$$2x+2y+3z \qquad 6xz+6yz+9z^2$$

8. Extract the square root of $a^2-ab+\frac{1}{4}b^2$.

By mere inspection we perceive that the root is $a-\frac{1}{2}b$; and the roots of examples 9 and 10 are equally obvious from inspection. The first term of the root of 9, is obviously $\frac{a}{b}$; and if the 2d term is not obvious, it can be discovered by dividing $-2$ by $\frac{2a}{b}$, which gives $-\frac{2}{1}\times\frac{b}{2a}=-\frac{b}{a}$.

11. The first term of the root is 1, and the second is $-2z$; hence, we have,

$$1-4z+10z^2-20z^3+25z^4-24z^5+16z^6(1-2z+3z^2-4z^3$$

$$\underline{1-4z+4z^2}$$

$$2-4z+3z^2 \qquad 6z^2-20z^3+25z^4$$

$$\underline{6z^2-12z^3+9z^4}$$

$$2-4z+6z^2-4z^3 \qquad -8z^3+16z^4-24z^5+16z^6$$

$$-8z^3+16z^4-24z^5+16z^6$$

12. $a^6 - 6a^5c + 15a^4c^2 - 2a^3c^3 + 15a^2c^4 - 6ac^5 + c^6$. This we observe to be the sixth power of $a - c$; hence, the square root of the given power must be the third power of $(a - c)$;

or, $a^3 - 3a^2c + 3ac^2 - c^3$, *Ans.*

13. $z^2 - 2z + 1 + 2zh - 2h + h^2$. Here we observe the square of $z$, of 1, and of $h$, and twice the product of $z$ and 1, of $z$ and $h$, and of 1 and $h$; hence, the square root must be $z + h - 1$, or, $1 - h - z$.

---

## SQUARE ROOT OF NUMBERS.

(**189**, page 190.)

To aid in extracting the square and cube roots in numbers, we should have the powers of numbers before us, as in the following table:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Numerals, | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Squares, | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 |
| Cubes, | 1 | 8 | 27 | 64 | 125 | 216 | 343 | 512 | 729 | 1000 |

3. Square root of $88,36(\overset{a}{90} + \overset{b}{4}$

$$\begin{array}{rl}
2a = 180 & 81 \\
\text{Second divisor, } 2a + b = 184 & 7\ 36 \\
 & 7\ 36
\end{array}$$

We have solved several examples in this article by another method. For another solution of this example, see page 100.

4. $10,69,29(\overset{a}{3}\overset{b}{2}\overset{c}{7}$, *Ans.*

$$\begin{array}{rl}
a^2 = & 9 \\
2a + b = 62 & 1\ 69 \\
2a + 2b = 64 & 1\ 24 \\
2a + 2b + c = 647 & 45\ 29 \\
 & 45\ 29
\end{array}$$

For another solution of this example, see page 100.

5.

| | | |
|---|---|---|
| | | 478,29,69)$\overset{a}{2}\overset{b}{1}\overset{c}{8}\overset{d}{7}$ |
| $a^2$ | | 4 |
| $2a + b$ | 41 | 78 |
| | | 41 |
| $2a + 2b + c$ | 428 | 37 29 |
| | | 34 24 |
| $2a + 2b + 2c + d$ | 4367 | 3 05 69 |
| | | 3 05 69 |

6.

| | | |
|---|---|---|
| | | 43,04,67,21($\overset{a}{6}\overset{b}{5}\overset{c}{6}\overset{d}{1}$ |
| $a^2$ | | 36 |
| $2a + b$ | 125 | 7 04 |
| | | 6 25 |
| $2a + 2b + c$ | 1306 | 79 67 |
| | | 78 36 |
| | 13121 | 131 21 |
| | | 131 21 |

7.

| | | |
|---|---|---|
| | | 3,87,42,04,89($\overset{a}{1}\overset{b}{9}\overset{c}{6}83$ |
| $a^2 =$ | | 3 61 |
| $2a + b$ | 386 | 26 42 |
| | | 23 16 |
| $2a + 2b + c$ | 3928 | 3 26 04 |
| | | 3 14 24 |
| | 39363 | 11 80 89 |
| | | 11 80 89 |

8. 12,09,99,62,25(34785

9

64 3 09

2 56

687 53 99

48 09

6948 5 90 62

5 55 84

69565 34 78 25

34 78 25

9. 65,96,03,86,56(81216

64

161 1 96

1 61

1622 35 03

32 44

16241 2 59 86

1 62 41

162426 97 45 56

97 45 56

For another solution of this example, see page 101.

10. 3,42,69,41,44(18512

3 24

365 18 69

18 25

3701 44 41

37 01

37022 7 40 44

7 40 44

11. 2,57,37,33,56,97,96(1604286

2 56

3204 1 37 33

1 28 16

32082 9 17 56

6 41 64

320848 2 75 92 97

2 56 67 84

3208566 19 25 13 96

19 25 13 96

12. 10.4976(3.24

9

62 1 49

1 24

644 2576

2576

13 3271.4207(57.19 +

| | |
|---|---|
| | 25 |
| 107 | 771 |
| | 749 |
| 1141 | 22 42 |
| | 11 41 |
| 11429 | 11 0107 |
| | 10 2861 |

14. 4795.25731(69.247 +

| | |
|---|---|
| | 36 |
| 129 | 1195 |
| | 1161 |
| 1382 | 34 25 |
| | 27 64 |
| 13844 | 6 6173 |
| | 5 5376 |
| 13848 | 1 079710 |

For another solution of this example, see page 101.

16. .00032754(.01809 +

| | |
|---|---|
| | 1 |
| 28 | 227 |
| | 224 |
| 3609 | 35400 |
| | 32481 |

17. .00103041(.0321

| | |
|---|---|
| | 9 |
| 62 | 130 |
| | 124 |
| 641 | 641 |

20. 7225(85

| | |
|---|---|
| | 64 |
| 165 | 825 |
| | 825 |

17689(133

| | |
|---|---|
| | 169 |
| 263 | 789 |
| | 789 |

$\frac{85}{133}$, *Ans.*

22. $\frac{98}{128} = \frac{49}{64} = \frac{7}{8}$, *Ans.*

23. $\frac{112}{175} = \frac{16}{25} = \frac{4}{5}$, *Ans.*

24. $\frac{3304}{7434} = \frac{4}{9} = \frac{2}{3}$.

25. $\frac{2704}{4225} = \frac{16}{25} = \frac{4}{5}$.

26. .75(.8660 +

| | |
|---|---|
| | 64 |
| 166 | 1100 |
| | 996 |
| 1726 | 10400 |
| | 10356 |
| 1732 | 4400 |

27. $\frac{7}{9}$ = .7777 + (.8819 +

| | |
|---|---|
| | 64 |
| 168 | 1377 |
| | 1344 |
| 1761 | 3377 |
| | 1761 |
| 17629 | 161677 |
| | 158661 |

**Note.** The square roots of numbers can be found, more or less approximately, by any person who can operate in division, by conceiving the square root to be, as it really is, one of two equal factors of the given number.

By assuming one factor, and dividing the number by it, we shall have the other corresponding factor; and if one of them is less than the root, the other will be greater, and the root will be very nearly midway between the two. In fact, it will be a very *little less than the half sum of the two factors.*

We will solve, by this method, a few examples in the last article, commencing at

( Page 190. )

3. Required, the square root of 8836. Here are two periods; hence, the root will consist of two digits. The root of the left-hand period is greater than 9, and consequently the root is more than 90. Suppose it to be 96.

96)8836(92
864
196
192

Thus we learn that the two factors, 96 and 92, will give a product a very little less than 8836. Therefore, the square root must be between 92 and 96. It is 94, *Ans.*

4. What is the square root of 106929? In other words, find two equal factors of 10,69,29. Here are three periods, and the square root of 10 is greater than 3, and consequently the whole root is greater than 300. Suppose it to be 320.

```
32|0)10692|9(334
      96
     ---
      109              334
       96              320
      ---             ----
       132            2)654
       128            -----
       ---             327, Ans.
        49/320
```

9.
```
81|000)6596038|656(81432
       648
       ---
       116             1st factor, 81000
        81             2d     "    81432
       ---                       --------
       350                       2)162432
       324                       --------
       ---                          81216, sq. root.
        263
        243
        ---
        208
```

13. What is the cube root of 3271.4207?

```
Assumed factor,   57)3271.4207(57.3933
                     285
                     ---
                      421       1st factor, 57.000
                      399       2d     "    57.3933
                      ---                  --------
                       224              2)114.3933
                       171                ---------
                       ---                57.19665, sq. root.
                        532
                        513
                        ---
                         190
                         171
                         ---
                         197
```

Thus we might operate with any number of examples.

## CUBE ROOT OF ALGEBRAIC EXPRESSIONS.

(**191**, page 194.)

NOTE. The cube of a monomial, or single term, as $a$, is $a^3$; the cube of $(a+b)$ is $a^3+3a^2b+3ab^2+b^3$; and the cube of the sum of any two quantities, or letters, is always in this form. Hence, if a quantity is not in this form, or cannot be reduced to this form, it is probable that it is a surd, or that no cube root of it exists. Also observe, that the cube of $a-b$ is $a^3-3a^2b+3ab^2-b^3$.

In either case, the first and last terms of the cube are cubes, The cube consists of 4 terms; in one case the terms in the cube are all plus, in the other case, alternately plus and minus.

If we should require the cube root of $a^3+2ab^3+ab^2+b^3$, we should recognize at once that the quantity is not a cube and of course that a cube root is impossible.

Hence, whenever the cube root of an algebraic expression is required, it is presumed that the quantity is a cube.

1. What is the cube root of $8+12a+6a^2+a^3$.

Here we have 4 terms; the first is 8, the cube of 2; the last is $a^3$, the cube of $a$. Hence, if a cube root exist, it must be $2+a$.

2. What is the cube root of $27a^3+108a^2+144a+64$?

Here again we have 4 terms, all plus; the first term is the cube of $3a$; and the last term is the cube of 4. Hence, again, $3a+4$, must be the cube root of the whole in case a root exists; and by cubing $3a+4$, we shall discover that the whole expression is a perfect cube.

3. What is the cube root of $a^3-6a^2x+12ax^2-8x^3$?

Here the number of terms is 4. The first term is a cube, and so is the last, and every alternate sign is minus. Therefore, $a-2x$ must be the cube root of this quantity.

We can find the second term of the root, by taking the second term of the cube $-6a^2x$, and dividing it by 3 times the square of the cube root of the first term. $3a^2) -6a^2x\ (-2x.$

Were there a remainder by this division, it would show that the expression was not a cube.

4. As the cube root of this polynomial is not obvious, we will apply the rule on page 194 of the text-book.

$$x^6 - 3x^5 + 5x^3 - 3x - 1\,(x^2 - x - 1$$
$$x^6$$

| | | | |
|---|---|---|---|
| | | $3x^4$ | $-3x^5 + 5x^3 - 3x$ |
| $3x^2 - x$ | $-3x^3 + x^2$ | $3x^4 - 3x^3 + x^2$ | $-3x^5 + 3x^4 - x^3$ |
| | | $3x^4 - 6x^3 + 3x^2$ | $-3x^4 + 6x^3 - 3x - 1$ |
| $3x^2 - 3x - 1$ | $-3x^2 + 3x + 1$ | $3x^4 - 6x^3 + 3x + 1$ | $-3x^4 + 6x^3 - 3x - 1$ |

7.

| | | | | |
|---|---|---|---|---|
| | | | $x^9+3x^8+6x^7+10x^6+12x^5+12x^4+10x^3+6x^2+3x+1$ | |
| | | | $x^9$ | Cube root. |
| | Trial divisor, | $3x^6$ | $3x^8+6x^7+10x^6$ | $(x^3+x^2+x+1$ |
| $3x^3+x^2$ | $3x^5+x^4$ | $3x^6+3x^5+x^4$ | $3x^8+3x^7+x^6$ | |
| | Trial divisor, | $3x^6+6x^5+3x^4$ | $3x^7+9x^6+12x^5+12x^4+10x^3$ | |
| $3x^3+3x^2+x$ | $3x^4+3x^3+x^2$ | $3x^6+6x^5+6x^4+3x^3+x$ | $3x^7+6x^6+6x^5+3x^4+x^3$ | |
| | Trial divisor, | $3x^6+6x^5+9x^4+6x^3+3x^2$ | $3x^6+6x^5+9x^4+9x^3+6x^2+3x+1$ | |
| $3x^3+3x^2+3x+1$ | $3x^3+3x^2+3x+1$ | $3x^6+6x^5+9x^4+9x^3+6x^2+3x+1$ | $3x^6+6x^5+9x^4+9x^3+6x^2+3x+1$ | |

8.

| | | | |
|---|---|---|---|
| | | | $a^9-3a^8+8a^6-6a^5-6a^4+8a^3-3a+1(a^3-a^2-a+1$ |
| | | | $a^9$ |
| | Trial divisor, | $3a^6$ | $-3a^8+8a^6-6a^5$ |
| $3a^3-a^2$ | $-3a^5+a^4$ | $3a^6-3a^5+a^4$ | $-3a^8+3a^7-a^6$ |
| | Trial divisor, | $3a^6-6a^5+3a^4$ | $-3a^7+9a^6-6a^5-6a^4$ |
| $3a^3-3a^2-a$ | $-3a^4+3a^3+a^2$ | $3a^6-6a^5+3a^3+a^2$ | $-3a^7+6a^6-3a^4+a^3$ |
| | Trial divisor, | $3a^6-6a^5-3a^4+6a^3+3a^2$ | $3a^6-6a^5-3a^4+9a^3-3a+1$ |
| $3a^3-3a^2-3a-1$ | $3a^3-3a^2-3a+1$ | $3a^6-6a^5-3a^4+9a^6-3a+1$ | $3a^6-6a^5-3a^4+9a^3-3a+1$ |

The same rule is to be applied when we extract the cube roots of numbers.

(195)

(193.)

3. 148,877(53

| | | | |
|---|---|---|---|
| | | $5^3 =$ | 125 |
| | Trial div. | 7500 | 23 877 |
| 153 | 459 | 7959 | 23 877 |

4. 571,787(83

| | | | |
|---|---|---|---|
| | | $8^3$ | 512 |
| | Trial div. | 19200 | 597 87 |
| 243 | 729 | 19929 | 597 87 |

5. 1,367,631(1.11

| | | | |
|---|---|---|---|
| | | $11^3$ | 1 331 |
| | Trial div. | 36300 | 36 631 |
| 331 | 331 | 36631 | 36 631 |

6. 2,048,383(127

| | | | |
|---|---|---|---|
| | | $a^3$ | 1 728 |
| | $3a^2 =$ | 43200 | 320 383 |
| 367 | 2569 | 45769 | 320 383 |

7. 16,581,375(255

| | | | |
|---|---|---|---|
| | | | 8 |
| | Trial div. | 1200 | 8 581 |
| 65 | 325 | 1525 | 7 625 |
| | Trial div. | 1875 | 956 375 |
| 755 | 3775 | 191275 | 956 375 |

(198)

8.

| | | | |
|---|---|---|---|
| | | | 44,361,864(354 |
| | | | 27 |
| | Trial div. | 2700 | 17 361 |
| 95 | 475 | 3175 | 15 875 |
| | Trial div. | 367500 | 1 486 864 |
| 1054 | 4216 | 371716 | 1 486 864 |

9.

| | | | |
|---|---|---|---|
| | | | 100,544,625(465 |
| | | | 64 |
| | Trial div. | 4800 | 36 544 |
| 126 | 756 | 5556 | 33 336 |
| | Trial div. | 634800 | 3 208 625 |
| 1385 | 6925 | 641725 | 3 208 625 |

10.

| | | | |
|---|---|---|---|
| | | | 12,358,435,328(2312 |
| | | | 8 |
| | Trial div. | 1200 | 4 358 |
| 63 | 189 | 1389 | 4 167 |
| | Trial div. | 158700 | 191 435 |
| 691 | 691 | 159391 | 159 391 |
| | Trial div. | 160083 | 32 044 328 |
| 6932 | 13864 | 16022164 | 32 044 328 |

11.

| | | | |
|---|---|---|---|
| | | | 999,700,029,999(9999 |
| | | | 729 |
| | Trial div. | 24300 | 270 700 |
| 279 | 2511 | 26811 | 241 299 |
| | Trial div. | 2940300 | 29 401 029 |
| 2979 | 26811 | 2967111 | 36 703 999 |
| | Trial div. | 299400300 | 2 697 030 999 |
| 29979 | 269811 | 299670111 | 2 697 030 999 |

( 198, 199 )

12. 2,456(13.49 +

2 197

| | | | |
|---|---|---|---|
| | Trial div. | 50700 | 259000 |
| 394 | 1576 | 52276 | 209104 |
| | Trial div. | 53868 | 49896000 |
| 4029 | 36261 | 5423061 | 48807549 |
| | | | 1088451 |

13. .004,019,679(.159

1

| | | | |
|---|---|---|---|
| | Trial div. | 300 | 3 019 |
| 35 | 175 | 475 | 2 375 |
| | | 675 | 644 679 |
| 459 | 4131 | 71631 | 644 679 |

14. 2,287.148(13.175 +

$a^3 = 2\ 197$

| | | | |
|---|---|---|---|
| | Trial div. | 50700 | 90 148 |
| 391 | 391 | 51091 | 51 091 |
| | | 5148300 | 39 057 000 |
| 3937 | 27559 | 5175859 | 36 231 113 |
| | | | 2 825 887 |

**Note.** When we know that a number is a complete cube, and the root less than 100, we can determine it by simple inspection; thus, in Example 8, what is the cube root of 148,877?

The superior, or left-hand, period is 148: the cube number next less is 125, the root of which is 5; hence, the root is between 50 and 60. The unit period is 877, the unit figure 7; and none of the cube numbers (single digits) have 7 for their unit figure except the cube of 3, which is 27; therefore, the unit figure in the root must be 3, and the whole root 53, *Ans.*

**4.** Required the cube root of 571,787. The cube root must be 83, because the cube of 8 is 512, the cube next less than 571; and the unit figure in the root must be 3, because the unit figure in the cube is 7.

Again, the number 366,048 is a cube: what is its cube root? *Ans.* 72. The teacher may multiply examples at pleasure, giving numbers whose cube roots are less than 100.

## (194.)

### CUBE ROOT OF NUMBERS BY EQUAL FACTORS.

The cube root of a number is one of three equal factors of that number. Having one factor of a number, we can obtain another by division; and having two factors, or the product of two, we can obtain a third factor by the same method.

Keeping these principles in view, any one who can multiply and divide numbers can extract the cube root more or less approximately. To illustrate this method, let us take Example 14, the one just solved; that is, let us demand one of the three equal factors of 2287.148.

The cube of 13 is 2187; hence, we know that each factor, or the cube root, must be between 13 and 14. The product of 13 by 14 is 182; and if we divide 2287.148 by 182, we shall have the third corresponding factor; thus,

```
182)2287.148(12.56674
    182
    ---
     467
     364
    ----
     1031
      910
     ----
      1214
      1092
      ----
       1228
       1092
       ----
        1360
        1274
        ----
         860
```

Three factors of the given number are thus found to be,

```
          12.56674
          13
          14
          --------
Sum, 3)39.56674
          --------
          13.18891
```

Whence, we learn that 13.18891 must be near in value to one of three equal factors of the number, or, in other words, to *one third part* of the number. One third of the sum of three unequal factors of any number we know is always a little greater than the cube root; and the error increases as the factors become more unequal, and diminishes as they become nearer and nearer equal; and, by repeating the operation, we can obtain the approximate cube root to any required degree of exactness. The cube root, in the present example, we know to be less than 13.188, and greater than 13.1; we will therefore assume it to be 13.16, $(13.16)^2 = 173.1856$.

```
173.1856)2287.148(13.2063
         1731 856
         --------
          555 2920
          519 5568
          --------
           35 73520
           34 63712
           --------
            1 0980800
            1 0391136
            ---------
               5896640
```

| | | |
|---|---|---|
| Thus, the 1st factor is | | 13.16 |
| 2d | " | 13.16 |
| 3d | " | 13.2063 |
| | | 3)39.5263 |
| Cube root, | | 13.1754 |

For another illustration, let us take Ex. 11, page 199 of the text-book:

What is the cube root of 999700029999?

We will assume 10000 as one of the factors; and as the factors must be equal, let the same number be another factor. The product of these is 100000000. Taking this for a divisor to the given number, the third factor must be $9997\frac{29999}{100000000}$. But we may reject the fraction, because the absolute quotient is always a little too large.

| | |
|---|---:|
| Whence, for one factor, we have | 10000 |
| For another, | 10000 |
| For the third, | 9997 |
| | 3)29997 |

Therefore, one of 3 equal factors must be 9999, *Ans.*

NOTE. When the given number is known to be a cube, this method is less troublesome than the direct, exact, and scientific method. We will illustrate by Example 9, from page 198.

Given 100,544,625, a cube, to find its root.

We know that the cube of 4 is 64, and of 5, 125. Hence, we know at once that the cube root must be between 400 and 500, because 100 is between 64 and 125, and we know that it must be nearer 500 than 400. Hence, we will suppose it to be 460.

```
(460)² = 211600)1005446|25(475
                 8464
                 ————
                 15904          1st factor, 460
                 14812          2d    "     460
                 —————          3d    "     475, fraction omitted.
                  10926                    ————
                  10580                   3)1395
                  —————                    ————
                    346     Three equivalent to 465, Ans.
```

## *Other Contracted Methods.*

The method given in the text-book, on pages 200 and 201, is sufficiently illustrated on those pages.

We will now show the application of French's method of binomial coefficients to the extraction of cube roots.

Any number may be represented by $a^3b$, and its cube root is $a\sqrt[3]{b}$. That is, any number contains some cube number as a factor. The other corresponding factor $b$ may, or may not, be a cube. If it is a cube, let it be represented by $n^3$. Then, the number itself will be $a^3n^3$, and its cube root $an$, *the product of the cube root of each factor taken separately.*

Thus, the number 3375 is a cube, because it is the product of the two cube numbers, 125 and 27. The cube root is $5 \times 3$ or 15.

The cube root of 29 can be found as follows:

Conceive 29 to be the product of two factors, one of which must be a complete cube, as 27. The other factor is 1.074 +.

The cube root of 27 is 3, and the cube root of (1+.074), is $(1+.074)^{\frac{1}{3}}$, and consequently, the cube root of 29 is $3(1+.074)^{\frac{1}{3}}$.

Let $a = 1$, and $b = .074$. Then $(1 + .074)^{\frac{1}{3}} = (a + b)^n$.

Now, $(a + b)^n$, is easily expanded into a series, by French's method for coefficients. $n = \frac{1}{3}$.

$$C_1 = a^n = 1^{\frac{1}{3}} = 1$$

$$C_2 = \frac{C_1 n b}{a} = \frac{1 \times \frac{1}{3} \times .074}{1} = .024\tfrac{2}{3}$$

$$C_3 = \frac{.024\frac{2}{3} \times (\frac{1}{3} - 1) \times .074}{2} = -(.024\tfrac{2}{3})^2$$

$$C_4 = \frac{-(024\frac{2}{3})^2 \times (\frac{1}{3} - 2) \times .074}{3} = (.024\tfrac{2}{3})^{\frac{5}{3}}$$

| | |
|---|---|
| 1st and 2d terms | = 1.02466666 |
| Less.— 3d term | = .00058244 |
| Algebraic sum of 3 terms | = 1.02408422 |
| + 4th term | = .00002394 |
| Sum of 4 terms | = 1.02410816 |
| Cube root of 27 | = 3 |
| Cube root of 29 | = 3.07232448 |

This result must be a trifle too large, because the 5 term will be minus. But we may take as many terms as we please,

and therefore, *by this method, we can directly determine the cube, or any other root of any number, to any required degree of exactness.*

1. What is the approximate cube root of 122?

Observe that $122 = 125 - 3 = 125\left(1 - \frac{3}{125}\right)$.

Or, $\qquad 122 = 125(1 - .024)$.

Whence the cube root of 122 is equal to $5(1 - .024)^{\frac{1}{3}}$.

This corresponds to $(a - b)^n$; $\quad a = 1,\ b = .024,\ n = \frac{1}{3}$.

Whence,

$$C_1 = \qquad\qquad 1$$

$$C_2 = 1 \times -\tfrac{1}{3} \times .024 \qquad = -.008$$

$$C_3 = \frac{-.008(\frac{1}{3} - 1) - .024}{2} \qquad = -.000064$$

$$C_4 = \frac{-.000064(\frac{1}{3} - 2) - .008}{3} \qquad = -.0000002844$$

$$C_5 = \frac{-0000001422(\frac{1}{3} - 3) - .008}{4} = -.0000000007684.$$

| | |
|---|---|
| From | 1 |
| Subtract sum of 4 terms, | .0080642851684 |
| Sum of the series, | .9919357148316 |
| Multiply by 5, | 5 |
| Cube root of 122, | 4.9596785741580 |

NOTE. This should be true as far as ten decimal places. The answer in the text-book is only approximate, corresponding to the first operation under that rule.

2. What is the cube root of 10?

$10 = 8 + 2 = 8(1 + \frac{1}{4})$. Cube root $= 2(1 + \frac{1}{4})^{\frac{1}{3}}$.

$$C_1 = \qquad\qquad 1$$

$$C_2 = 1 \times \tfrac{1}{3} \times \tfrac{1}{4} = \quad \tfrac{1}{12} \quad = .083$$

$$C_3 = \frac{\frac{1}{12}(\frac{1}{3}-1)\frac{1}{4}}{2} = -\frac{1}{144} = -.00694$$

$$C_4 = \frac{-\frac{1}{144}(\frac{1}{3}-2)\frac{1}{4}}{3} = +.000966$$

$$C_5 = \frac{.000966(\frac{1}{3}-3)\frac{1}{4}}{4} = -.000161$$

$$C_6 = \frac{-.000161(\frac{1}{3}-4)\frac{1}{4}}{5} = +.0000209514$$

$$C_7 = \frac{.0000209514(\frac{1}{3}-5)\frac{1}{4}}{6} = -.000004076.$$

| | + Plus terms. | − Minus terms. |
|---|---|---|
| | 1.0833333333 | .0069444444 |
| | .0009660000 | .0001610000 |
| | 209514 | 40740 |
| | 1.0843202847 | .0071095184 |
| Subtract | .0071095184 | |
| | 1.0772107663 | |
| Multiply by | 2, or the cube root of 8. | |
| | 2.1544215326 | |

This is a trifle less than the true root, because the next term of the series would be plus.

8. Required the approximate cube root of 720.

$$720 = 729 - 9 = 729(1 - \tfrac{9}{729}) = 729(1 - \tfrac{1}{81}).$$

Cube root of $720 = 9(1 - \frac{1}{81})^{\frac{1}{3}}$. Here $a = 1$, $b = -\frac{1}{81}$, $n = \frac{1}{3}$.

$$C_1 =$$

$$C_2 = 1 \times \tfrac{1}{3} \times -\tfrac{1}{81} = -\tfrac{1}{243}$$

$$C_3 = \frac{-\frac{1}{243}(\frac{1}{3}-1)-\frac{1}{81}}{2} = -(\tfrac{1}{243})^2$$

$$C_4 = \frac{1 - (\frac{1}{243})^2(\frac{1}{3} - 2) - \frac{1}{81}}{3} \qquad = -(\tfrac{1}{243})^3 \tfrac{5}{9}$$

$$C_5 = \frac{-(\frac{1}{243})^3 \frac{5}{3}(\frac{1}{3} - 3) - \frac{1}{81}}{4} \qquad = -(\tfrac{1}{243})^4 \tfrac{10}{3}$$

$-\ \frac{1}{243} =$ .004115226338

$-(\frac{1}{243})^2 =$ .000016935089

$-(\frac{1}{243})^3 = .0000000696917 \times \frac{5}{9} =$ .116152

$-(\frac{1}{243})^4 = .0000000002869 \times \frac{10}{3} =$ .956

Subtract .004132278535

From 1.

Difference, 0.995867721475

9

Cube root of 720 = 8.95280949328

4. **What is the approximate cube root of 345?**

$345 = 343 + 2 = 343(1 + \frac{2}{343}).\ \sqrt[3]{345} = 7 \times (1 + \frac{2}{343})^{\frac{1}{3}}.$

$$C_1 = \qquad 1$$

$$C_2 = 1 \times \frac{1}{3} \times \frac{2}{343} \qquad = \frac{2}{343 \times 3};$$

$$C_3 = \frac{\frac{2}{1029}\left(\frac{1}{3} - 1\right)\frac{2}{343}}{2} \qquad = -\frac{4}{(343)^2 9};$$

$$C_4 = \frac{\frac{4}{(343)^2 \times 9}\left(\frac{1}{3} - 2\right)\frac{2}{343}}{3} = \frac{40}{(343)^3 81}.$$

| | |
|---|---|
| 1st and 2d terms, | 1.0019436346 |
| 4th term, +, | .00000001221 |
| Sum of plus terms, | 1.00194364681 |
| 3d term, minus, —, | .00000376919 |
| | 1.00193987762 |
| Cube root of 343 is | 7 |
| Cube root of 345, | 7.01357914334 True to 10 places. |

5. What is the approximate cube root of 520?

$$\sqrt[3]{520} = \sqrt[3]{512 + 8} = \sqrt[3]{512}\sqrt[3]{1 + \tfrac{8}{512}} = 8(1 + \tfrac{1}{64})^{\frac{1}{3}}.$$

$$C_1 = \qquad 1,$$

$$C_2 = 1 - \tfrac{1}{3} \times \tfrac{1}{64} \qquad = \tfrac{1}{192};$$

$$C_3 = \frac{\frac{1}{192}(\frac{1}{3} - 1)\frac{1}{64}}{2} \qquad = -(\tfrac{1}{192})^2;$$

$$C_4 = -(\tfrac{1}{192})^2\frac{(\frac{1}{3} - 2)}{3}\tfrac{1}{64} = (\tfrac{1}{192})^3\,\tfrac{5}{3};$$

$$C_5 = \tfrac{5}{3}(\tfrac{1}{192})^3\frac{(\frac{1}{3} - 3)}{4}\tfrac{1}{64} \qquad = -(\tfrac{1}{192})^4\,\tfrac{25}{36}.$$

$\frac{1}{192} = .00520833$, $(\frac{1}{192})^2 = .00002712673$,
$(\frac{1}{193})^3 = .0000001413$, $(\frac{1}{192})^4 = .0000000007$; whence,

| | |
|---|---|
| 1st and 2d terms of the series, | 1.0052083333 |
| 4th term, plus, | .0000002355 |
| | 1.0052085688 |

| | | | |
|---|---|---|---|
| 3d term, | .00002712673 | | |
| 5th term, | .00000000043 | | |
| | .00002712716 | Subtract | .0000271272 |
| Sum of the series, | | | 1.0051814416 |
| | | | 8 |
| Cube root required, | | | 8.041451328 |

6. What is the cube root of 65?

$$\sqrt[3]{65} = \sqrt[3]{64 + 1} = 4\sqrt[3]{1 + \tfrac{1}{64}}.$$

Here the values of $a$, $b$, and $n$, are the same as in the preceding example. Therefore, the value of the series will be the same.

| | |
|---|---|
| Hence, | 1.00518144 |
| | 4 |
| Cube root required, | 4.02072576 |

NOTE.—We will now apply this method to a complete cube in whole numbers.

Suppose the cube root to be 2800. The cube of this is

$$\begin{array}{l} 21952|000000)22069|810125(1.00532 \\ \quad 21952 \\ \hline \quad\quad 117810 \\ \quad\quad 109760 \\ \hline \quad\quad\quad 70501 \end{array}$$

Here $a = 1$, $b = .0053$, $n = \frac{1}{3}$.

$$C_1 =$$

$$C_2 = 1 \times \tfrac{1}{3} \times 0053 \qquad = .00177 +$$

$$C_3 = \frac{.00177(\frac{1}{3} - 1).0053}{2} = -(.00177)^2$$

$$C^4 = \frac{-(.00177)^2(\frac{1}{3} - 2).0053}{3} = (.00177)^3 \times \tfrac{5}{3}$$

| | |
|---|---|
| Sum of the plus terms, | 1.0017700092 |
| Minus terms, | .0000031329 |
| Sum of the series, | 1.0017668763 |
| Multiply by | 2800 |
| | 80141350104 |
| | 2003.5337526 |
| Cube root, | 2804.95725 |

This is a small fraction less than the true cube root, *as it ought to be*, because the decimal, .00177, was not carried out to its full value. The true cube root is 2805, and our result is within a small fraction of that number. We know the unit figure in the root must be 5, because the unit figure in the cube is 5, and the number was given as a complete cube.

9. Here the number is given as a perfect cube. The unit figure in the cube is 8. Therefore the unit figure in the root

must be 2, and the cube itself must be divisible by 8, without a remainder, and the quotient must be a cube; whence,

$$\begin{array}{r} 8)41135081408 \\ \hline 8)5141885176 \text{ a cube,} \\ \hline 642.735.647 \end{array}$$

After the first division, the unit figure is 6, an even number; therefore, the whole is divisible by 2, and possibly by 8. Certainly by 8, or by 216, the cube of 6. We try 8, and find the second complete quotient. This must also be a cube. This last cube contains only three periods; and the superior one is 642. The greatest cube in this is 512, the root being 8. Three times the square of 8 is 192, the trial divisor for the next figure.

$$\begin{array}{lr} \text{From} & 6427 \\ \text{Subtract} & 512 \\ \hline 192) & 1307(6 \\ & 1142 \end{array}$$

The next figure is therefore 6, and as the unit figure of the cube is 7, the unit figure of the root must be 3.

| | |
|---|---|
| That is, the cube root of the cube 642735647, is | 863 |
| Multiply this root by 2, twice, or by 4; thus, | 4 |
| Whence the root of the given cube must be | 3452 |

10. The number 125.525.735.343 is a cube. What is its root? There are 4 periods, therefore there must be 4 places in the cube root. The superior period is 125, and its cube root is 5. The second figure is obviously 0, and the unit figure 7, therefore we would at once try 507. That is, cube this number, and thus discover whether it be the true root or not.

The true root might have some significant figure in the place of 10, for aught we could discover by observation.

## REDUCTION OF RADICALS.

(**200**, page 205.)

Omit the factor without the radical, for a moment, until the one under the radical is reduced. First separate the quantity under the radical into two factors, one of which is a complete power. Thus, $\sqrt{x^3y^2 - x^2y^3}$, is equal to $\sqrt{x^2y^2 \times (x-y)}$.

Extract the root of the complete power, and bring it without the radical, and we shall have $xy\sqrt{x-y}$. Multiply this result by the factor omitted, $xy$, and we shall have $x^2y^2\sqrt{x-y}$ for the reduced expression. In like manner, perform all other examples in this Case.

10.
$$\begin{array}{r} \sqrt{36bc^2 \times 2ac} = 6bc\sqrt{2ac} \\ 4 \\ \hline 24bc\sqrt{2ac}, \textit{ Ans.} \end{array}$$

11.
$$\begin{array}{r} \sqrt{49a^4x^2 \times 3ay} = 7a^2x\sqrt{3ay} \\ 2a \\ \hline 14a^3x\sqrt{3ay}, \textit{ Ans.} \end{array}$$

12.
$$\begin{array}{r} \sqrt[3]{125 \times x} = 5\sqrt[3]{x} \\ 5 \\ \hline 25\sqrt[3]{x}, \textit{ Ans.} \end{array}$$

13.
$$\begin{array}{r} (32c^{10} \times ac)^{\frac{1}{5}} = 2c^2(ac^{\frac{1}{5}} \\ 2c \\ \hline 4c^3(ac)^{\frac{1}{5}}, \textit{ Ans.} \end{array}$$

14.
$$\begin{array}{r} \sqrt{(a^2 - 2ab + b^2)a} = (a-b)\sqrt{a} \\ (a+b) \\ \hline (a^2 - b^2)\sqrt{a}, \textit{ Ans.} \end{array}$$

15. $\sqrt{(a^2+2ab+b^2)b} = (a+b)\sqrt{b},$
$$\frac{(a-b)}{(a^2-b^2)\sqrt{b},}\ Ans.$$

16. $d\sqrt{(x^2-2xy+y^2)xy} = d(x-y)\sqrt{xy},$ *Ans.*

17. $\sqrt{36x^2 \times 5xy} = 6x\sqrt{5xy},$ *Ans.*

18. $(8x^3y^3 \times 3x^2z)^{\frac{1}{3}} = 2xy(3x^2z)^{\frac{1}{3}},$ *Ans.*

19. $(27m^9 \times 2a)^{\frac{1}{3}} = 3m^3(2a)^{\frac{1}{3}},$ *Ans.*

20. $\sqrt[3]{a^6z^3(a-b)} = a^2z(a-b)^{\frac{1}{3}},$ *Ans.*

(**201**, page 206.)

5. $(a+cx)$ must be raised to the 4th power, and put under the index for the 4th root, the value then being the same, but form and appearance very different.
$$\left((a+cx)^4\right)^{\frac{1}{4}} = (a^4+4a^3cx+6a^2c^2x^2+4ac^3x^3+c^4x^4)^{\frac{1}{4}}.$$

6. Square each factor; $(a^3\sqrt{c})^{2\cdot\frac{1}{2}} = \sqrt{a^6c},$ *Ans.*

7. Cube each factor; $27a^3 \times 2a^4x$, and $\sqrt[3]{54a^7x},$ *Ans.*

8. Cube each factor, $(2a-c)^3 \times 4$; then,
$$(32a^3-48a^2c+24ac^2-4c^3)^{\frac{1}{3}},\ Ans.$$

9. Raise each factor to its 5th power.

$4c^3$ to 5th power is $1024c^{15}$; $ac$ is the 5th power of the other factor. Product, $(1024ac^{15})^{\frac{1}{5}}$, *Ans.*

10. $\left(a^4x^8 \times (a+bxy)\right)^{\frac{1}{4}} = \sqrt[4]{a^5x^8+a^4bx^9y},$ *Ans.*

(**205**, **206**)

11. $ab + x$, squared, is $(ab + x)^2$. The square of the other factor is $a^2b^2 - 2abx + x^2$; or, it is $(ab - x)^2$.

Whence, $(ab + x)^2 \times (ab - x)^2 = (a^2b^2 - x^2)^2$;

hence, $\sqrt{a^4b^4 - 2a^2b^2x^2 + x^4}$, *Ans.*

12. $\left((a^4 - 2a^2b^2 + b^4) \times a\right)^{\frac{1}{2}} = (a^5 - 2a^3b^2 + ab^4)^{\frac{1}{2}}$.

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## ADDITION.

( **203**, page 208. )

Radicals can be united into one sum, or one term, provided the same quantity is under the radical, to form a *unit of addition;* or, provided we can reduce the several terms to the *same unit of addition.*

3. $3\sqrt{3a^2x}$ and $a\sqrt{48x}$ can be united in one sum, because the first term is equal to $3a\sqrt{3x}$, and the second is the same as $a\sqrt{16 \times 3x} = 4a\sqrt{3x}$. Here $\sqrt{3x}$ is the *unit* of addition. The sum is obviously $7a\sqrt{3x}$; and the difference is $a\sqrt{3x}$.

4. $\sqrt{80m} = \sqrt{16 \times 5m} = 4\sqrt{5m}$, and $\sqrt{125m} = \sqrt{25 \times 5m} = 5\sqrt{5m}$. Sum, $9\sqrt{5m}$, *Ans.*

5. $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$. $\sqrt{128} = \sqrt{64 \times 2} = 8\sqrt{2}$. $\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$. Sum is $16\sqrt{2}$; the square root of 2 being the unit of addition.

6. 1st term, $xy\sqrt{3a}$; 2d, $3xy\sqrt{3a}$; 3d, $2xy\sqrt{3a}$. Sum, $6xy\sqrt{3a}$.

7. 1st term, $a\sqrt{2xy}$; 2d term, $b\sqrt{2xy}$. Sum, $(a + b)\sqrt{2xy}$, *Ans.*

8. $a\sqrt{x-y}+2a\sqrt{x-y}=3a\sqrt{x-y}$, *Ans.*

9. $\sqrt{80a^2b^2} = \sqrt{16a^2b^2\times 5} = 4ab\sqrt{5}$. $\sqrt{245a^8b^6} = \sqrt{49a^8b^6\times 5} = 7a^4b^3\sqrt{5}$. Sum, $(4ab+7a^4b^3)\sqrt{5}$.

10. $3\sqrt{3a^2x}=3a\sqrt{3}$. $\sqrt{12a^2x}=2a\sqrt{3x}$. $\sqrt{3b^2x}= b\sqrt{3x}$. Sum, $(5a+b)\sqrt{3x}$.

11. $\sqrt{2a^4}=a^2\sqrt{2}$; $2\sqrt{2a^2b^2}=2ab\sqrt{2}$; $\sqrt{2b^4}=b^2\sqrt{2}$. Sum, $(a^2+2ab+b^2)\sqrt{2}$; or, $(a+b)^2\sqrt{2}$, *Ans.*

12. Sum is obviously $2\sqrt[3]{a^2}+5\sqrt[3]{a^2}$; or, $7\sqrt[3]{a^2}$, *Ans.*

13. $\sqrt[3]{270a^3m}=\sqrt[3]{27a^3\times 10m}=3a(10m)^{\frac{1}{3}}$; $\sqrt[3]{1250b^3m} = \sqrt[3]{125b^3\times 10m}=5b(10m)^{\frac{1}{3}}$. Sum, $(3a+5b)\,(10m)^{\frac{1}{3}}$, *Ans.*

14. $\sqrt[3]{x^7y}=\sqrt[3]{x^6\times xy}=x^2\sqrt[3]{xy}$. $\sqrt[3]{8x^3y^3\times xy} = 2xy\sqrt[3]{xy}$. $\sqrt[3]{xy^7}=\sqrt[3]{y^6\times xy}=y^2\sqrt[3]{xy}$. Hence, $(x+y)^2\sqrt[3]{xy}$.

15. 1st term $=\frac{3}{4}\sqrt{a}$; and 2d term $=\frac{1}{4}\sqrt{a}$. Hence, $\sqrt{a}$.

16. 1st term $a\sqrt{b}$, 2d term $b\sqrt{a}$. Sum, $a\sqrt{b}+b\sqrt{a}$.

17. 1st term, $x\sqrt{m}$; 2d term, $x\sqrt{n}$. Sum, $x(\sqrt{m}+\sqrt{n})$. Here $x$ is the unit of addition.

18. 1st term, $2\times 2a\sqrt{b}=4a\sqrt{b}$; 2d term, $6a\sqrt{b}$. Sum, $10a\sqrt{b}$.

19. 1st term, $\sqrt[3]{a^3x^3(x-y)} = ax\sqrt[3]{x-y}$; 2d term, $ax\sqrt[3]{(x-y)}$. Sum, $2ax(x-y)^{\frac{1}{3}}$.

## SUBTRACTION.

( **204**, page 210. )

3. $\sqrt{162x^4y} = \sqrt{81x^4 \times 2y} = 9x^2\sqrt{2y}$

$4\sqrt{8x^4y} = 4\sqrt{4x^4 \times 2y} = 8x^2\sqrt{2y}$

$x^2\sqrt{2y}$, *Ans.*

4. $\sqrt{20a^4b^3y} = \sqrt{4a^4b^2 \times 5by} = 2a^2b\sqrt{5by}$

$\sqrt{5a^4yb^3} = \sqrt{a^4b^2 \times 5by} = a^2b\sqrt{5by}$

$a^2b\sqrt{5by}$, *Ans.*

5. $3\sqrt[3]{128a^3bc} = 3\sqrt[3]{64a^3 \times 2bc} = 12a\sqrt[3]{2bc}$

$4a\sqrt[3]{16bc} = 4a\sqrt[3]{8 \times 2bc} = 8a\sqrt[3]{2bc}$

$4a\sqrt[3]{2bc}$, *Ans.*

6. $\sqrt[3]{375a^4b} = \sqrt[3]{125a^3 \times 3ab} = 5a\sqrt[3]{ab}$

$\sqrt[3]{24ab^4} = \sqrt[3]{8b^3 \times 3ab} = 2b\sqrt[3]{3ab}$

$(5a - 2b)\sqrt[3]{3ab}$, *Ans.*

7. $3(16a^8 \times ab^2)^{\frac{1}{4}} = 6a^2(ab^2)^{\frac{1}{4}}$

$2a(a^4 \times ab^2)^{\frac{1}{4}} = 2a^2(ab^2)^{\frac{1}{4}}$

$4a^2(ab^2)^{\frac{1}{4}}$, *Ans.*

8. $\sqrt{3a^2c + 6abc + 3b^2c} = \sqrt{(a^2 + 2ab + b^2)3c} = (a + b)$,

$\sqrt{3c}$, $\sqrt{12b^2c} = \sqrt{4b^2 \times 3c} = 2b\sqrt{3c}$

From $(a + b)$, take $2b$, and $(a - b)\sqrt{3c}$, *Ans.*

9. $\sqrt{2a^3c^2} = \sqrt{a^2c^2 \times 2a} = ac\sqrt{2a}$

$\sqrt{2bc^2} = c\sqrt{2b}$

$ac\sqrt{2a} - c\sqrt{2b}$, *Ans.*

Here we have no common unit of subtraction.

10. $\sqrt{a^3 - a^2b} = \sqrt{a^2(a-b)} = a\sqrt{a-b}$
$\sqrt{ab^2 - b^3} = \sqrt{b^2(a-b)} = b\sqrt{a-b}$

$(a-b)\sqrt{a-b}$, *Ans.*

12. $6\sqrt[3]{32} = 6\sqrt[3]{8 \times 4} = 12\sqrt[3]{4}$
$6\sqrt[3]{\frac{4}{27}} = 6\sqrt[3]{\frac{1}{27} \times 4} = \frac{6}{3}\sqrt[3]{4}$

$10\sqrt[3]{4}$, *Ans.*

13. $\left.\begin{array}{l} 2\sqrt[5]{32a^5 \times a^4} = 4a\sqrt[5]{a^4} \\ 4\sqrt[5]{a^5b^4} \quad\;\; = 4a\sqrt[5]{b^4} \end{array}\right\}$ *Diff.* $4a(a^{\frac{4}{5}} - b^{\frac{4}{5}})$, *Ans.*

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## MULTIPLICATION.

( **205**, page 211. )

3. $5\sqrt{a^2cm} = 5a\sqrt{cm}$, *Ans.*

4. $12\sqrt{100x^2y} = 120x\sqrt{y}$, *Ans.*

5. $2\sqrt[3]{27x^3yz} = 6x\sqrt[3]{yz}$, *Ans.*

6. $6\sqrt[3]{8 \times 7} = 12\sqrt[3]{7}$, *Ans.*

7. $6\sqrt{9} = 6 \times 3 = 18$, *Ans.*

8. $12\sqrt{16} = 12 \times 4 = 48$, *Ans.*

9. $\sqrt{6 \times 150} = \sqrt{900} = 30$, *Ans.*

10. Prod. $\sqrt{\frac{3}{16}} = \frac{1}{4}\sqrt{3}$, *Ans.*

11. Prod. $a\sqrt{b} + b$, obvious.

( **210, 211** )

12. Prod. of sum by diff. = diff. of squares, $x^2 - y$, *Ans.*

13. Diff. of squares is $m - n$, *Ans.*

14. The square of $\sqrt{a} + \sqrt{c}$ is $a + 2\sqrt{ac} + c$, *Ans.*

15. Result obvious. $ac(bd)^{\frac{1}{2}}$, *Ans.*

16. Result, $2c(3a^3b^2d)^{\frac{1}{3}} = 2ac(3b^2d)^{\frac{1}{3}}$, *Ans.*

17. Prod. $(x^4y^4)^{\frac{1}{4}} = xy$, *Ans.*.

( **206**, page 212. )

2. $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$.

We multiply like quantities by the addition of their exponents. We therefore add $\frac{1}{2}$ and $\frac{1}{3} = \frac{5}{6}$. Hence, the product must be $a^{\frac{5}{6}}$, *Ans.*

4. Let P represent the required product.

Then, $$P = \sqrt{\frac{1}{2}} \times \left(\frac{3}{8}\right)^{\frac{1}{3}}.$$

Squaring both members, $$P^2 = \frac{1}{2} \times \left(\frac{3}{8}\right)^{\frac{2}{3}}.$$

Cubing both members, $$P^6 = \frac{1}{8} \times \left(\frac{3}{8}\right)^2 = \frac{9}{8 \times 64}.$$

Take the 6th root, and $$P = \frac{1}{2} \times \left(\frac{9}{8}\right)^{\frac{1}{6}}, \textit{ Ans.}$$

5. $$P = \sqrt{2} \times (2)^{\frac{1}{5}}.$$

By squaring, $$P^2 = 2 \times (2)^{\frac{2}{5}} = 2^{\frac{7}{5}}.$$

Extracting square root, and $P = 2^{\frac{7}{10}} = (2^7)^{\frac{1}{10}} = (128)^{\frac{1}{10}}$.

( **211, 212** )

6. Same as 2, by writing $(a+b)$ in place of $a$;

whence, $(a+b)^{\frac{5}{6}}$, *Ans.*

7. $$P = 4\sqrt{a} \times 3b(d+x)^{\frac{1}{n}},$$

By squaring, $$P^2 = (12b)^2 a(d+x)^{\frac{2}{n}}.$$

Raise to the $n$th power, and

$$P^{2n} = (12b)^{2n}a^n(d+x)^2 = (12b)^{2n} \times (a^n(d+x)^2).$$

Taking the $2n$th root, and $P = 12b(a^n(d+x)^2)^{\frac{1}{2n}}$, *Ans.*

8. $$P = 2a^2(b)^{\frac{1}{3}}(y)^{\frac{1}{6}}(c+x^2)^{\frac{1}{n}}.$$

Raise to the 6th power, and we shall have,

$$P^6 = 2^6a^{12}b^2y(c+x^2)^{\frac{6}{n}}.$$

To the $n$th power, $P^{6n} = (2a^2)^{6n} \times b^{2n}y^n(c+x^2)^6$.

Take $6n$ root, and $P = 2a^2 \times (b^{2n}y^n(c+x^2)^6)^{\frac{1}{6n}}$.

9. $$P = \left(\frac{a}{b}\right)^{\frac{1}{2}}\left(\frac{a^3}{x}\right)^{\frac{1}{4}}.$$

Raise to the 4th power, and

$$P^4 = \frac{a^2}{b^2} \times \frac{a^3}{x} = \frac{a^5}{b^2x}.$$

Take the 4th root, and $P = \left(\frac{a^5}{b^2x}\right)^{\frac{1}{4}}$.

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## DIVISION.

( **207**, page 214. )

3. The quotient is obviously $\sqrt{25a^2y}$; reduced, $5a\sqrt{y}$, *Ans.*

4. Quotient, $2\sqrt{100m^3}$; reduced, $20m\sqrt{m}$, *Ans.*

5. Quotient, $\sqrt{20}$; reduced, $2\sqrt{5}$, *Ans.*

6. Quotient, $\sqrt{9} = 3$, *Ans.*

7. Quotient, $4\sqrt{12}$; reduced, $4\sqrt{4 \times 3} = 8\sqrt{3}$, *Ans.*

8. In any division, the product of the divisor and quotient is equal to the dividend. Therefore, let Q represent the quotient;

$$\sqrt{15}\,Q = 3\sqrt{10}.$$

By squaring, $\quad 15\,Q^2 = 9 \times 10.$

Dividing by $3 \times 5$, and $\quad Q^2 = 3 \times 2 = 6,$

$$Q = \sqrt{6}, \textit{ Ans.}$$

9. $$Q^3ab = a^4b^2c, \text{ or } Q^3 = a^3bc,$$
$$Q = a(bc)$$

10. Dividing term by term, and

$$4(x^4y^6)^{\frac{1}{4}} = 4xy \times \sqrt{y}, \textit{ Ans.} \text{ reduced.}$$

11. By division, $(b^2cd^3)^{\frac{1}{3}} = d(b^2c)^{\frac{1}{3}}$, *Ans.*

12. Quotient, $\sqrt{a^2 - a^3b} = \sqrt{a^2(1 - ab)} = a\sqrt{1 - ab}.$

15. $$Q\sqrt{\frac{a}{b}} = \sqrt{\frac{x}{y}}$$

$$Q^2\frac{a}{b} = \frac{x}{y}. \quad Q^2 = \frac{bx}{ay} = \frac{abxy}{a^2y^2}.$$

Square root, $\quad Q = \frac{1}{ay}\sqrt{abxy}.$

( **208**, page 215. )

Let Q represent the quotient in each of the following examples, thus forming equations.

2. $$(xy)^{1}Q = (ax^2)^{\frac{1}{3}}.$$

Raise each factor to the 21st power; then,

$$x^3y^3Q^{21} = a^7x^{14}$$

$$Q^{21} = \frac{a^7x^{11}}{y^3}. \quad Q = \left(\frac{a^7x^{11}}{y^3}\right)^{\frac{1}{21}}, \textit{ Ans.} \text{ reduced.}$$

( **214, 215** )

3. $\sqrt[3]{xQ} = x^{\frac{1}{5}}$. Whence, $xQ^3 = x^{\frac{3}{5}}$

$$x^5Q^{15} = x^3 \quad Q^{15} = \frac{1}{x^2} = x^{-2}$$

$$Q = x^{-\frac{2}{15}}, \textit{ Ans.}$$

4. $\sqrt{5}Q = 30$. By squaring $5Q^2 = 30 \times 30$

$$Q^2 = 6 \times 30 = 36 \times 5$$

$$Q = 6\sqrt{5}, \textit{ Ans.}$$

5. $5(a + c)^{\frac{1}{4}}Q = 10x(a + c)^{\frac{2}{3}}$

$$(a + c)Q^4 = (2x)^4(a + c)^{\frac{8}{3}}$$

$$Q^4 = (2x)^4(a + c)^{\frac{5}{3}}$$

$$Q = 2x(a + c)^{\frac{5}{12}}, \textit{ Ans.}$$

6. Omit the common factor $(a + x)$; then,

$$(m + y)^{\frac{n}{m}}Q = (a - x)\,(m + y)^{\frac{1}{n}}.$$

Raise each member to $mn$th power, and

$$(m + y)^{2n}Q^{mn} = (a - x)^{mn}(m + y)^m$$

$$Q^{mn} = (a - x)^{mn}(m + y)^{m-2n}$$

$$Q = (a - x)\,(m + y)^{\frac{m-2n}{mn}}.$$

7. $\left(\frac{3}{8}\right)^{\frac{1}{3}}Q = \left(\frac{2}{5}\right)^{\frac{1}{2}}$

$$\left(\frac{3}{8}\right)^{\frac{2}{3}}Q^2 = \frac{2}{5}$$

Cubing $\frac{9}{64}Q^6 = \frac{8}{125}$

$$9Q^6 = \frac{8 \times 8 \times 8}{5 \times 5 \times 5}$$

$$Q^6 = \frac{64 \times 8}{125 \times 9}$$

$$Q = 2\sqrt{\tfrac{8}{1125}}, \textit{ Ans.}$$

8. $$\frac{(a^2x)^{\frac{1}{3}}}{\sqrt{x}}Q = \frac{(ax^3)^{\frac{1}{2}}}{(ax)^{\frac{1}{3}}}$$

Clearing of fractions, $(a^3x^2)^{\frac{1}{3}}Q = (ax^4)^{\frac{1}{2}}$

Cubing, $a^3x^2Q^3 = (ax^4)^{\frac{3}{2}}$

Squaring, $a^6x^4Q^6 = a^3x^{12}$

$a^3Q^6 = x^8 = x^6x^2$

Cube root, $aQ^2 = x^2 \times x^{\frac{2}{3}}$

Whence, $Q^2 = x^2 \times \frac{x^{\frac{2}{3}}}{a}$, and $Q = \frac{x \times x^{\frac{1}{3}}}{\sqrt{a}}$, *Ans.*

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## SIMPLE EQUATIONS

### CONTAINING RADICAL QUANTITIES.

(**211**, page 218.)

5. By squaring each member, we obtain,

$$4 + \sqrt{x-2} = 9, \text{ or } \sqrt{x-2} = 5.$$

By squaring again, and transposing 2, we have, $x = 27$, *Ans.*

6. By transposition, $x + 2 = \sqrt{x^2 + 6}$.

By squaring, and omitting $x^2$ in each member,

$4x + 4 = 6$; whence, $4x = 2$, and $x = \frac{1}{2}$, *Ans.*

7. Transposing $x$, and then squaring each member, produces, $x^2 - 7 = 49 - 14x + x^2$;

whence, $14x = 49 + 7$.

Dividing by 7, $2x = 7 + 1 = 8$, and $x = 4$, *Ans.*

8. By squaring, $x + 12 = 4 + 4\sqrt{x} + x$.

By omitting $x$ in each member, and dividing by 4,

$3 = 1 + \sqrt{x}$; or $\sqrt{x} = 2$, and $x = 4$, *Ans.*

9. By squaring, we have,

$$4 + 4\sqrt{3x} + 3x = 5x + 4;$$

$$\text{whence, } 4\sqrt{3x} = 2x, \text{ or } 2\sqrt{3x} = x.$$

By squaring, $4 \times 3x = x^2$.

Dividing by $x$, and $x = 3 \times 4 = 12$.

10. By squaring, $x^2 + 4x + 4 = 4 + x\sqrt{64 + x^2}$.

Dropping 4 from each member, and dividing by $x$, we shall have, $x + 4 = \sqrt{64 + x^2}$.

By squaring again, and omitting $x^2$ in each member,

$$8x + 16 = 64,$$

$$x + 2 = 8; \text{ or } x = 6, \textit{ Ans.}$$

11. Divide each member of each separate term by $\sqrt{x}$;

then, $\sqrt{x} - \frac{1}{2} = \sqrt{x - 1}$.

By squaring, $x - \sqrt{x} + \frac{1}{4} = x - 1$.

By transposition, $\frac{5}{4} = \sqrt{x}$. By squaring, $x = \frac{25}{16}$, *Ans.*

12. $\sqrt{x - 2a} = a - \sqrt{x}$.

By squaring, $x - 2a = a^2 - 2a\sqrt{x} + x$.

By transposition, $2a\sqrt{x} = a^2 + 2a$.

Dividing by $a$, $2\sqrt{x} = a + 2 = 18$; (A)

$\sqrt{x} = 9$, and $x = 81$, *Ans.*

13. By clearing of fractions, $3 + x = 6$; $x = 3$, *Ans.*

14. Same solution as in Example 11, as far as to Equation (A), in which let $a = 8$.

Then, $2\sqrt{x} = 10$; $\sqrt{x} = 5$; and $x = 25$, *Ans.*

15. By squaring, $x + 3a = 4a$; whence, $x = a$, *Ans.*

16. Dividing numerator by denominator, in the 2d member, and

$$\sqrt{cx+a} = \sqrt{2a}.$$

By squaring, $cx + a = 2a.$

Reducing, $cx = a$; and $x = \frac{a}{c}$, *Ans.*

17. By squaring,

$$x + 2a = 2a + 2\sqrt{2a}(x-2a)^{\frac{1}{2}} + x - 2a;$$

whence, $2a = 2\sqrt{2a}(x-2a)^{\frac{1}{2}}.$

Divide by $\sqrt{2a}$, and

$$\sqrt{2a} = 2(x-2a)^{\frac{1}{2}}.$$

By squaring, $2a = 4(x-2a) = 4x - 8a$;

whence, $4x = 10a$; $2x = 5a$; and $x = \frac{1}{2}(5a)$, *Ans.*

18. Multiply each term by $\sqrt{m^2 - \sqrt{x}}$. Then,

$$1 = m^2 - x\sqrt{m^2 - x}$$

By transposition, $x\sqrt{m^2-x} = m^2 - 1.$

By squaring, $m^2x - x = (m^2-1)^2$;

or, $(m^2-1)x = (m^2-1)^2.$

Dividing, $x = m^2 - 1$, *Ans.*

19. Clearing of fractions, and

$$(\sqrt{x}+28)(\sqrt{x}+6) = (\sqrt{x}+38)(\sqrt{x}+4).$$

By actual multiplication, we obtain

$$x + 28\sqrt{x} + 6\sqrt{x} + 6 \times 28 = x + 38\sqrt{x} + 4\sqrt{x} + 4 \times 38.$$

Dropping $x + 34\sqrt{x}$, we have

$$6 \times 28 = 8\sqrt{x} + 4 \times 38.$$

Dividing by 4, $6 \times 8 = 2\sqrt{x} + 38$;

reducing, $4 = \sqrt{x}$; or, $\sqrt{x} = 2$;

and $x = 4$, *Ans.*

20. Given $x^3 - y^3 = 56$, and $xy(x-y) = 16$, to find the value of $x$.

Multiply the 2d equation by $-3$; then,

$$-3x^2y + 3xy^2 = -48 \quad (1)$$

Add the first equation, $$x^3 - y^3 = 56 \quad (2)$$

Sum is, $$x^3 - 3x^2y + 3xy^2 - y^3 = 8 \quad (3)$$

Each member of this equation is a cube, and the cube root is

$$x - y = 2 \quad (4)$$

Dividing (2) by (4) and

$$x^2 + xy + y^2 = 28 \quad (5)$$

Dividing the 2d of the given equations by (4), and

$$xy = 8 \quad (6)$$

Add equations (5) and (6), and

$$x^2 + 2xy + y^2 = 36 \quad (7)$$

Square root, $$x + y = 6 \quad (8)$$

Adding (4), $$2x = 8$$

Whence, $$x = 4; \text{ and } y = 2.$$

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## QUADRATIC EQUATIONS.

(**216**, page 222.)

7. Clear of fractions, and

$$3x^2 + 5 - 3x^2 + 5 = 10;$$

or, $$10 = 10; \text{ and } x = 0, \textit{ Ans.}$$

8. Multiply by 15, and

$$9x + 2x^2 = 9x + 18$$

$$2x^2 = 18;$$

whence, $$x^2 = 9; \text{ and } x = \pm 3, \textit{ Ans.}$$

(**219 - 222**)

9. Clear of fractions, and

$$6x^2 - 10 = 4x^2 - 4$$

$$2x^2 = 6;\ \text{and}\ x = \sqrt{3},\ \textit{Ans.}$$

10. By transposition, $(2\frac{3}{4})x^2 = 539;$

or, $11x^2 = 539 \times 4$

$$x^2 = 49 \times 4$$

$$x = \pm 7 \times 2 = \pm 14,\ \textit{Ans.}$$

11. Clearing of fractions, $c^2x^2 - a^2c^2 = a^2x^2$

Transposing and factoring, $(c^2 - a^2)x^2 = a^2c^2$

Dividing and extracting square root, $x = \pm \dfrac{ac}{\sqrt{c^2 - a^2}}.$

12. Clearing of fractions,

$$ax^2(a - 2) = 1 - x^2$$

or, $a^2x^2 - 2ax^2 + x^2 = 1$

Factoring, $(a^2 - 2a + 1)x^2 = 1$

Square root, $(a - 1)x = \pm 1$

And, $x = \pm \dfrac{1}{a - 1}$, *Ans.*

13. By squaring, $x^2 - 5 = \dfrac{4x^2}{9}$

$$5x^2 = 9 \times 5$$

$$x^2 = 9;\ \text{and}\ x = \pm 3,\ \textit{Ans.}$$

14. By squaring, $\dfrac{20x^2 - 9}{4x} = x;$

whence, $16x^2 = 9;\ 4x = \pm 3;$ and $x = \pm \frac{3}{4}.$

15. By transposing and reducing,

$$10 = \sqrt{x^2 - 144};$$

whence, $x^2 = 144;$ and $x = \pm 12,$ *Ans.*

16. Clearing of fractions, $\sqrt{x^2 - 9} = 2\sqrt{10}$.

Squaring, and $\quad x^2 - 9 = 40$;

whence, $\quad x = \pm 7$.

17. By squaring, $x^2(a^2 + x^2) = a^4 - 2a^2x^2 + x^4$.

Reducing, $\quad 3a^2x^2 = a^4$,

$$x^2 = \frac{a^2}{3} \quad x = \pm a\sqrt{\tfrac{1}{3}},\ \textit{Ans.}$$

18. By squaring, $\quad x^2 - a^2 = a^2m - a^2$;

$$x^2 = a^2m,\ x = \pm a\sqrt{m}.$$

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## SOLUTION OF PROBLEMS

### PRODUCING PURE QUADRATIC EQUATIONS.

( **217**, page 225. )

4. Let $28 + x =$ one number, and $28 - x =$ the other;

then, $\quad (28 + x)(28 - x)$, or $\overline{28}^2 - x^2 = 640$;

or, $\quad x^2 = \overline{28}^2 - 640 = 784 - 640 = 144$,

$$x = \pm 12.$$

Whence, $28 + 12 = 40$, one number, and $28 - 12 = 16$, the other.

5. Let $12x$ represent the number; then, $4x = \frac{1}{3}$ of the number, and $3x = \frac{1}{4}$ of it.

$$12x^2 = 108,\ x^2 = 9,\ x = \pm 3,\ 12x = 36,\ \textit{Ans.}$$

Or, let $x =$ the number;

then, $\quad \dfrac{x}{3} \times \dfrac{x}{4} = \dfrac{x^2}{12} = 108 = 9 \times 12$;

whence, $\quad x^2 = 9 \times 144$, or $x = 3 \times 12 = 36$, *Ans.*

6. Let $x =$ the number;

then, $x^2 + 18 = \frac{x^2}{2} + \frac{61}{2}$;

$2x^2 + 36 = x^2 + 61$;

$x^2 = 25$, and $x = \pm 5$, *Ans.*

8. Let $4x =$ the greater, and $3x =$ the less;

then, $16x^2 - 9x^2 = 7x^2 = 28$;

$x^2 = 4$, and $x = 2$.

9. Let $x =$ the less number, and $16x =$ the greater number; then, $\frac{\text{Greater, } 16x}{\text{Less, } x} = 16$, and $16x^2 = 144$.

Square root, $4x = 12$,

$$\left.\begin{aligned} x &= 3, \text{ less,} \\ \text{and } 16x &= 48, \text{ greater,} \end{aligned}\right\} \textit{Ans.}$$

10. Let $9x =$ the length of the lot, and $5x =$ the breadth; $45x^2 = 405$; $x^2 = 9$; $x = 3$; $9x = 27$, and $5x = 15$.

11. Let $2x =$ the difference, and $9x =$ the greater; then, $7x =$ the less number.

$81x^2 - 49x^2 = 32x^2 = 128$; $16x^2 = 64$; $4x = 8$; $x = 2$.

The numbers are 18 and 14, *Ans.*

12. The ratio of $\frac{1}{2}$ to $\frac{2}{3}$ is the same as 3 to 4; hence, let $4x =$ one, and $3x$ the other number.

$9x^2 + 16x^2 = 25x^2 = 225$;

$5x = 15$, and $x = 3$.

13. Let $6x =$ the length of the lot; and $5x =$ the breadth. Then, $30x^2$ is the whole lot. $\frac{1}{6}$ of $30x^2 = 5x^2$ for garden, leaves $25x^2$.

$25x^2 = 625$; whence, $x^2 = 25$, $x = 5$. $6x = 30$, length, *Ans.*

14. Let $x =$ the age of the younger.

Then, $(94 + x)(94 - x) = 8512$;

or, $8836 - x^2 = 8512$;

whence, $x^2 = 324$, and $x = 18$, *Ans.*

15. Let $x =$ the number.

Then, $9\sqrt{x^2 + 11} - 4 = 50$

$\sqrt{x^2 + 11} = 6$

$x^2 + 11 = 36$. $x = 5$, *Ans.*

16. Let $x =$ the sum gained.

Whence, $320 : x :: 5x : 2500$;

or, $320 : x :: x : 500$

$x^2 = 160000$. $x = 400$, *Ans.*

17. Let $x =$ the number sought.

Then, by the conditions, we must have

$$8 - \frac{x^2}{4} = 4; \text{ or, } \frac{x^2}{4} = 4;$$

whence, $\frac{x}{2} = \pm 2$, and $x = \pm 4$, *Ans.*

19. Let $8x =$ the length of the field, and $5x =$ the breadth.

Then, $\frac{40x^2}{160} = \frac{x^2}{4} =$ the number of acres.

$\frac{x^2}{4} \times 8x =$ the cost in dollars.

The number of rods round the field is $26x$.

The cost of the field in dollars was, then, $26x \times 13$;

whence, $\frac{x^2}{4} \times 8x = 26x \times 13$;

or, $2x^2 = 26 \times 13$;

or, $x^2 = 13 \times 13$, $x = 13$.

Therefore, $13 \times 8 = 104$ rods in length, and $13 \times 5 = 65$ rods in breadth, *Ans.*

20. Let $2x$, $3x$, and $5x$, represent the numbers.

Product, $30x^3 = 108 \times 10x$;

whence, $x^2 = 36,\ x = 6,\ 2x = 12$, &c.

21. Let $x =$ the greater number; and $14 - x =$ the less.

Then, $\dfrac{x}{14 - x} : \dfrac{14 - x}{x} : : 16 : 9.$

Multiply the first couplet by $x$, then by $(14 - x)$,

and $x^2 : (14 - x)^2 : : 16 : 9$

Square root, $x : 14 - x : : 4 : 3$

Equation, $3x = 14 \times 4 - 4x$

Reduce $7x = 14 \times 4,\ x = 2 \times 4 = 8.$

22. Let $6 + x =$ one number, and $6 - x =$ the other; then, their sum is 12; their product is $36 - x^2$; it is also 35;

whence, $35 = 36 - x^2$,

$x^2 = 1,\ x = 1$, and 7 and 5 are the numbers.

23. Let $x + 3$, and $x - 3$ represent the numbers;

$$\begin{array}{rl} x^2 + 6x + 9 = & \text{the square of one,} \\ x^2 - 6x + 9 = & \text{the square of the other.} \\ \hline \text{Sum,}\quad 2x^2 + 18 = 50; & \end{array}$$

$x^2 + 9 = 25;\ x^2 = 16;\ x = 4;$

whence, the numbers are 7 and 1.

24. Let $x + 4$, and $x - 4$ represent the numbers;

then, $x^2 - 16 = 240;\ x^2 = 256;\ x = 16;$

$16 - 4 = 12$, and $16 + 4 = 20$, *Ans.*

## AFFECTED QUADRATICS.

### (**220**, page 231.)

**NOTE. Some teachers solve all equations by the formules on page 230; that is,** $x = -a + \sqrt{b + a^2}$, **one root,**

**or,** $x = -a - \sqrt{b + a^2}$, **the other root.**

**We will use these to solve equations from 2 to 10, page 231;** $2a =$ **the coefficient of** $x$, **and** $b$ **is the absolute term.**

2. Here, $2a = 4$;

whence, $a^2 = 4$; $b = 96$; $\sqrt{b + a^2} = 10$;

whence, $x = -2 \pm 10 = 8$, or $-12$, *Ans.*

3. Here, $2a = -4$, $a^2 = 4$, $b = 45$,

$$\sqrt{b + a^2} = \sqrt{49} = \pm 7;$$

whence, $x = +2 \pm 7 = 9$, or $-5$, *Ans.*

4. $2a = -7$, $b = 8$,

$$\sqrt{b + a^2} = \sqrt{8 + \frac{49}{4}} = \sqrt{\frac{81}{4}} = \pm \frac{9}{2}.$$

$$x = \frac{7}{2} \pm \frac{9}{2} = 8, \text{ or } -1, \textit{ Ans.}$$

8. Here, $2a = -15$, $b = -54$, $a^2 = \frac{225}{4}$, $b = -\frac{216}{4}$;

$\sqrt{b + a^2} = \sqrt{\frac{9}{4}} = \pm \frac{3}{2}$; $x = \frac{15}{2} \pm \frac{3}{2} = 9$, or 6, *Ans.*

9. Here, $2a = -\frac{2}{3}$, $a^2 = \frac{1}{9}$, $b = \frac{399}{9}$,

$$\sqrt{b + a^2} = \sqrt{\frac{400}{9}} = \pm \frac{20}{3};$$

whence, $x = \frac{1}{3} \pm \frac{20}{3} = 7, -\frac{19}{3}$, *Ans.*

10. $2a=-\frac{5}{6},\ a=-\frac{5}{12},\ a^2=\frac{25}{144},\ b=\frac{24}{144},$

$$\sqrt{a^2+b}=\sqrt{\frac{49}{144}}=\pm\frac{7}{12};$$

whence, $x=\frac{5}{12}\pm\frac{7}{12}=1;$ or $-\frac{1}{6}$, *Ans.*

11. Here, $2a=-\frac{a}{b},\ a=-\frac{a}{2b},\ a^2=\frac{a^2}{4b^2},\ b=\frac{c}{d},$

$$\sqrt{b+a^2}=\left(\frac{c}{d}+\frac{a^2}{4b^2}\right)^{\frac{1}{2}};\ \text{whence, } x=\&c.$$

12. By transposition and division, we obtain,

$$x^2-10x=-24.$$

Here, $\sqrt{a^2+b}=\sqrt{1}=\pm 1;$

whence, $x=5\pm 1=6$, or 4, *Ans.*

13. Transposition and division will give $x^2-16x=-55$
$2a=-16,\ a=-8,\ \sqrt{a^2+b}=\sqrt{64-55}=\sqrt{9}=\pm 3;$
whence, $x=8\pm 3=11$, or 5, *Ans.*

14. Transposition and division will give $x^2+2x=168;$
$x=-1\pm\sqrt{169}=-1\pm 13=12$, or $-14$, *Ans.*

15. Transposition and division will give $x^2+22x=-117;$
$x=-11\pm\sqrt{121-117}=-11\pm 2=-9-13$, *Ans.*

16. Here $2a=-\frac{9}{2},\ b=-2,\ a^2=\frac{81}{16},$

$$\sqrt{a^2+b}=\sqrt{\frac{81}{16}-\frac{32}{16}}=\sqrt{\frac{49}{16}}=\pm\frac{7}{4};$$

whence, $x=\frac{9}{4}\pm\frac{7}{4}=4$, or $\frac{1}{2}$, *Ans.*

17. Dropping 1, and doubling, then

$$-x=8-\frac{72}{x+2}$$

$$-x^2-2x=8x+16-72.$$

Transposing and changing signs, then

$$x^2+10x=56$$

$$x^2+10x+25=81$$

$$x=-5\pm 9=4, \text{ or } -14, \textit{ Ans.}$$

18. $9-3x^2=2x-76$

$3x^2+2x=85$, by transposing and changing signs.

$$x^2+\frac{2}{3}x=\frac{85}{3}$$

Complete square, $x^2+\frac{2}{3}x+\frac{1}{9}=\frac{255}{9}+\frac{1}{9}=\frac{256}{9}$

Extract root, $x+\frac{1}{3}=\pm\frac{16}{3}$. Whence, $x=5$, or $-\frac{17}{3}$.

19. Multiplying by 10, produces

$$10x^2+5x=4x^2-2x+13;$$

whence, $6x^2+7x=13.$

NOTES. 1.—Observe that $6+7=13$, therefore, $x$ can not be greater, nor less than 1, as *a positive number*. If $x$ were greater than 1, $6x^2$ would be greater than 6, and $7x$ would be greater than 7, and the sum of the two quantities would be greater than 13. But they are not greater than 13, therefore, $x$ can not be greater than 1.

Now if $x$ were part of 1, its square would be but a part of 1, and the 6 parts, each less than 1, would be less than 6, and $7x$ would be less than 7. But less than 6, added to less than 7, can not make 13; therefore, $x$ cannot be less than 1. Now, as $x$ can not be greater than 1, nor less than 1, it must be 1.

The same is true in any other equation; that is, *if the sum of the coefficients be the same in each member of the equation, then one value of the unknown symbol is* 1. But this is not the method of solution intended.

Divide by 6, then, $x^2+\frac{7}{6}x=\frac{13}{6}=\frac{7}{6}+1.$

Assuming $2a=\frac{7}{6}$, and we have $x^2+2ax=2a+1.$

Add $a^2$ to each, $x^2+2ax+a^2=a^2+2a+1$.

By evolution, $x+a=\pm(a+1)$;

whence, $x=1$, or $-2a-1-\frac{7}{6}-\frac{6}{6}=-\frac{13}{6}=-2\frac{1}{6}$.

2. Study this solution, it is a special artifice. The rule for completing this square was not set aside, but strictly observed. The artifice was in the employment of the symbol $a$.

20. Transposing, and clearing of fractions, and we obtain

$$x^2-4x=32$$

Whence, $x=2\pm\sqrt{32+4}=2\pm6=8$, or $-4$, *Ans.*

21. Dropping $7\frac{1}{3}$ from each member, and we have

$$\frac{x^2}{2}-\frac{x}{3}=\frac{1}{2}+\frac{1}{8}$$

$$x^2-\frac{2x}{3}=1\frac{1}{4}=\frac{5}{4}$$

$$x^2-\frac{2}{3}x+\frac{1}{9}=\frac{1}{9}+\frac{5}{4}=\frac{49}{36}$$

$$x=\frac{1}{3}\pm\frac{7}{6}=\frac{9}{6}, \text{ or } -\frac{5}{6}, \textit{ Ans.}$$

22. Omitting equals in each member, and

$$\frac{x^2}{4}-\frac{1}{4}=\frac{2x}{3}$$

By transposition, and multiplication,

$$x^2-\frac{8x}{3}=9$$

$$x^2-\frac{8x}{3}+\frac{16}{9}=\frac{16}{9}+\frac{9}{9}=\frac{25}{9}$$

$$x-\frac{4}{3}=\pm\frac{5}{3};\ x=3 \text{ or } -\frac{1}{3}.$$

23. Clearing of fractions, and we find

$$2x^2+x=x^2+3x+8x+24$$

$$x^2-10x=24$$

Whence, $x=5\pm\sqrt{49}=5\pm7=12$, or $-2$.

24. Transposing (1), and clearing of fractions, then we have

$$x^2-4x+2=x-4$$

$$x^2-5x=-6$$

$$x^2-5x+\frac{25}{4}=\frac{25}{4}-\frac{24}{4}=\frac{1}{4}$$

Square root, $$x-\frac{5}{2}=\pm\frac{1}{2}$$

$$x=\frac{6}{2}, \text{ or } 3, \text{ or } 2, \textit{ Ans.}$$

25. Clearing of fractions, and

$$22x-x^2-132+6x=300-20x.$$

Transposing, and changing signs, and

$$x^2-48x=-432$$

$$x^2-48x+(24)^2=576-432=144$$

Square root, $$x-24=\pm 12$$

$$x=36, \text{ or } 12, \textit{ Ans.}$$

26. Observe that the value of the 2d member is $\frac{1}{3}$.

Then, $$\frac{2x-7}{x-1}=\frac{1}{3};$$

or, $$6x-21=x-1$$

$$5x=20;\ x=4, \textit{ Ans.}$$

The other value does not appear in this solution, and ought not, because it is a simple equation.

28. By transposition,

$$x^2+(2d-4c)x=4cd-3c^2-d^2$$

Add $d^2-4cd+4c^2=d^2-4cd+4c^2$, to complete squares.

Root, $x+(d-2c)=\sqrt{c^2}=\pm c.$

whence, $x=3c-d$; or, $(c-d)$, *Ans.*

( **221**, page 235. )

Reduction of equations in the form

$$ax^2 + bx \qquad = c$$

Multiply by $4a$, $\quad 4a^2x^2 + 4abx \qquad = 4ac$

Complete square, $4a^2x^2 + 4abx + b^2 = 4ac + b^2$

Square root, $\quad 2ax + b = \sqrt{4ac + b^2}$

$$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}.$$

When $a = 1$, the preceding formula becomes

$$x = \frac{-b \pm \sqrt{4c + b^2}}{2}.$$

When $b$ is even, represent it by $2b^1$. Then this last formula becomes $x = -b^1 \pm \sqrt{c + \frac{b^2}{4}}$, the formula already used.

3. Here $a = 2$, $b = -5$, and $c = 117$;

$$4ac = 117 \times 8 = 936;$$

$$\sqrt{4ac + b^2} = \sqrt{936 + 25} = \sqrt{961} = \pm 31;$$

whence, $\quad x = \dfrac{+5 \pm 31}{4} = 9$, or $-6\frac{1}{2}$, *Ans.*

4. Here $a = 3$, $b = -5$, and $c = 28$;

$$4ac = 28 \times 12 = 336;$$

whence, $x = \dfrac{5 \pm \sqrt{336 + 25}}{6} = \dfrac{5 \pm 19}{6} = 4$, or $-\dfrac{7}{3}$, *Ans.*

5. Here $a = 3$, $b = -1$, $c = 70$, and $4ac = 840$;

$\sqrt{4ac + b^2} = \sqrt{841} = \pm 29$; $x = \dfrac{1 \pm 29}{6} = 5$, or $-\dfrac{14}{3}$

6. Here $a = 5$, $b = 4$, $c = 273$, $4ac = 5460$, $b^2 = 16$;

$\sqrt{4ac + b^2} = \sqrt{5476} = \pm 74$; $x = \dfrac{-4 \pm 74}{10} = 7$, or $-7\frac{4}{5}$.

7. $2x^2 + 3x = 65$; $16x^2 + (\;) + 9 = 65 \times 8 + 9 = 529$;

$$4x + 3 = \pm 23;\; x = 5, \text{ or } -\tfrac{13}{2}.$$

( **232 - 235** )

8. $$12(3x^2) + (\ ) + 25 = 42 \times 12 + 25 = 529;$$
$$6x + 5 = \pm 23; \text{ whence, } x = 3, \text{ or } -\tfrac{14}{3}.$$

9. $$8x^2 - 7x = 165,$$
and, $$32(8x^2) - (\ ) + 49 = 165 \times 32 + 49 = 5329;$$
$$16x - 7 = \pm 73; \text{ whence, } 16x = 80, \text{ or } -66;$$
$$\text{and } x = 5, \text{ or } -4\tfrac{1}{8}, \textit{ Ans.}$$

10. $$10x^2 - 8x = 312;$$
$$5x^2 - 4x = 156.$$
Put $x = \frac{y}{5}$; then, $5x^2 = \frac{y^2}{5}$;

and, $$\frac{y^2}{5} - \frac{4y}{5} = 156;$$
$$y^2 - 4y = 780;$$
$$y^2 - 4y + 4 = 784;$$
$$y - 2 = \pm 28; \ y = 30, \text{ or } -26;$$
whence, $$x = 6, \text{ or } -\tfrac{26}{5} = -5\tfrac{1}{5}.$$

Here, $a = 5$, $b = 4$, $c = 156$, and, $4ac = 3120$;
$$\sqrt{4ac + b^2} = \sqrt{3136} = \pm 56;$$
$$x = \frac{4 \pm 56}{10} = 6, \text{ or } -5\tfrac{1}{2}.$$

11. Here, $a = 3$, $b = 2$, and $c = 4$;
$$4ac = 48, \ \sqrt{4ac + b^2} = \sqrt{52};$$
whence, $$x = \frac{-2 \pm 2\sqrt{13}}{6} = -\frac{1}{3} \pm \frac{1}{3}\sqrt{13}.$$

12. Here, $a = 5$, $b = 7$, $c = 7$, and $4ac = 140$;
$$\sqrt{4ac + b^2} = \sqrt{189} = \sqrt{9 \times 21} = \pm 3\sqrt{21};$$
whence, $$x = -\frac{7}{10} \pm \frac{3}{10}\sqrt{21}.$$

13. Dividing by 4, and $\frac{60}{x} + \frac{1}{10} = \frac{54}{x-15}$;

$$(600 + x)(x - 15) = 540x.$$

Product, $600x + x^2 - 15{,}600 - 15x = 540x.$

Reduced, $x^2 + 45x = 45 \times 200.$

Square complete, $4x^2 + (\ ) + \overline{45}^2 = 180 \times 200 + \overline{45}^2 = 38025.$

Root, $2x + 45 = \pm 195;$

whence, $x = 75, \text{ or } -120.$

14. Divide each member by $x + 12$;

then, $1 = \frac{x}{5}$, or $x = 5$, *Ans*, one root.

But we divided by $(x + 12)$; then, this binomial factor contains a root of the equation, as the learner will see after his investigation of the theory of Equations in the University Algebra.

Therefore, $x + 12 = 0$, and $x = -12$, the other root.

15. Complete the square; multiply by 4, &c., and

$$4x^2 - (\ ) + 25 = 25 - 8 = 17;$$

$$2x - 5 = \sqrt{17} = \pm 4.12315;$$

whence, $2x = 9.12315, \text{ or } + .87685;$

and $x = 4.56157, \text{ or } + .43342.$

16. Multiply by 8; $16x^2 - (\ ) + 9 = 96 + 9 = 105.$

Square root, $4x - 3 = \sqrt{105} = \pm 10.2469;$

whence, $4x = 13.2469, \text{ or } -7.2469;$

$x = 3.3117, \text{ or } -1.8117.$

17. Here $a = 3$, $b = -1$, $c = 1$; whence, $4ac = 12.$

$\sqrt{4ac + b^2} = \sqrt{13}$. $x = \frac{1 \pm \sqrt{13}}{6} = .7675, \text{ or } -.4342.$

18. $$4x^2 - 4x + 1 = 5.$$

Square root, $2x - 1 = \sqrt{5}$, and $x = \frac{1 \pm \sqrt{5}}{2}$.

19. Here $a = 4$, $b = 3$, $c = 5$, and $4ac = 80$,

$$\sqrt{4ac + b^2} = \sqrt{89}, \quad x = \frac{-3 \pm \sqrt{89}}{8};$$

$$\text{or, } x = \frac{-3 \pm 9.434}{8} = \frac{6.434}{8} = .804 +$$

$$\text{or, } \frac{-12.434}{8} = -1.554.$$

20. Multiply by 4, and we have

$$4x^2 - (\ ) + 49 = 49 - 44 = 5;$$

$$2x - 7 = \sqrt{5} = \pm 223606.8;$$

whence, $x = 4.618034$, or $2.381966$.

**Note.** The pupil might provide himself with the square roots of 2, 3, 4, 5, &c., as far as 10 or 20; this would save considerable labor. Square roots, as far as 30, are in the 4th column, on page 293 of the text-book. The corresponding powers are in the 3d column.

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## HIGHER EQUATIONS IN THE QUADRATIC FORM.

(222, page 237.)

4. Place $y = x^2$; then, $y^2 = x^4$, and the equation becomes

$$y^2 - 3y = 550.$$

Here $a = 1$, $b = -3$, and $c = 550$. $4ac \pm b^2 = 2209$.

$$\sqrt{4ac + b^2} = \sqrt{2209} = \pm 47;$$

whence, $y = \frac{3 \pm 47}{2} = 25$, or $-22$.

That is, $x^2 = 25$; whence, $x = \pm 5$, or $x = \pm \sqrt{-22}$.

Here we have 4 roots, 5 and $-5$, $+\sqrt{-22}$ and $-\sqrt{-22}$;

two of them are imaginary, *because there is no square root to a negative number.*

That is, the given equation contains $x^4$; hence, there are 4 values to $x$.

5. Assume $y = \sqrt{x}$; then, $y^2 = x$, and the equation becomes

$$3y^2 - y = 44.$$

Here $a = 3$, $b = -1$, $c = 44$, and $4ac = 528$;

$$\sqrt{4ac + b^2} = \sqrt{529} = \pm 23;$$

$$y = \frac{1 \pm 23}{6} = 4, \text{ or } -\frac{11}{3}.$$

But $x = y^2 = 16$, or $\frac{121}{9} = 13\frac{4}{9}$.

6. Here let $2n = 7$; then, $2n + 1 = 8$; and the given equation becomes

$$x^6 - 2nx^3 = 2n + 1.$$

Add $n^2$ to each member; then,

$$x^6 - 2nx^3 + n^2 = n^2 + 2n + 1.$$

Square root, $x^3 - n = \pm (n + 1)$;

whence, $x^3 = 2n + 1$, or $-1$.

That is, $x^3 = 8$, or $-1$.

Cube root, $x = 2$, or $-1$.

7. Add 9 to each member, to complete the squares; then,

$$x^6 - 6x^3 + 9 = 576.$$

Square root, $x^3 - 3 = \pm 24$

$x^3 = 27$, or $-21$

$x = 3$, or $(-21)^{\frac{1}{3}} = -2.758 +$.

## POLYNOMIALS UNDER QUADRATIC FORMS.

( **223**, page 239. )

*Place the lowest power of the polynomial equal to a single letter*, and the form will be familiar.

3. Let $y^2 + 2y = P$ (1). Then, we have

$P^2 + 4P = 96$, a common quadratic in relation to $P$.

Square complete, $P^2 + 4P + 4 = 100$

$$P + 2 = \pm 10$$

$$P = 8, \text{ or } -12.$$

Whence, $y^2 + 2y = 8, \text{ or } -12$

Square complete, $y^2 + 2y + 1 = 9, \text{ or } -11$

Square root, $y + 1 = \pm 3; \text{ or } \pm \sqrt{-11}$

$$y = 2, \text{ or } -4, \text{ or } \pm \sqrt{-11}, \text{ 4 roots.}$$

4. Place $\sqrt{10 + x} = P$;

then, $P^2 - P = 12$

$$4P^2 - 4P + 1 = 49$$

$$2P - 1 = \pm 7$$

$$P = 4, \text{ or } -3.$$

That is, $\sqrt{10 + x} = 4, \text{ or } -3$

$$10 + x = 16, \text{ or } 9;$$

whence, $x = 6, \text{ or } -1$

5. Place $\frac{6}{y} + y = P$;

then, $P^2 + P + \frac{1}{4} = 30 + \frac{1}{4} = \frac{121}{4}$.

Square root, $P + \frac{1}{2} = \pm \frac{11}{2}$; $P = 5, \text{ or } -6$;

whence, $\frac{6}{y} + y = 5, \text{ or } -6$

$6 + y^2 = 5y, \text{ or } 6 + y^2 = -6y$

$y^2 - 5y = -6; \quad y^2 + 6y = -6$

$4y^2 - (\ ) + 25 = 25 - 24;$

and $y^2 + 6y + 9 = 3$

$2y - 5 = \pm 1; \quad y + 3 = \pm\sqrt{3}$

$y = 3 \text{ or } 2; \text{ or, } y = -3 \pm \sqrt{3}.$

6. Assume $(x + 12)^{\frac{1}{4}} = P.$

Then, $(x + 12)^{\frac{1}{2}} = P^2$, and the given equation becomes, $P^2 + P = 6$

$4P^2 + 4P + 1 = 25$

$2P + 1 = \pm 5$

$P = 2, \text{ or } -3.$

That is, $(x + 12)^{\frac{1}{4}} = 2, \text{ or } -3;$

4th power, $x + 12 = 16, \text{ or } +81;$

whence, $x = 4, \text{ or } 69.$

7. Assume $\sqrt{2x^2 + 3x + 9} = P.$

$P^2 - 5P = 6$

$4P^2 + (\ ) + 25 = 25 + 24 = 49$

$2P - 5 = \pm 7$

$P = 6, \text{ or } -1.$

That is, $2x^2 + 3x + 9 = P^2 = 36, \text{ or } 1;$

$2x^2 + 3x = 27, \text{ or } -8.$

Here, $a = 2,\ b = 3,\ c = 27,\ 4ac = 216$

$\sqrt{4ac + b^2} = \sqrt{225} = \pm 15$

$x = \frac{-3 \pm 15}{4} = 3, \text{ or } -4\frac{1}{2}.$

If $2x^2 + 3x = -8$

$x = -\frac{3}{4} \pm \frac{1}{4}\sqrt{-55}.$

8. Assume $(x + a)^{\frac{1}{4}} = P$;

and $(x + a)^{\frac{1}{2}} = P^2$.

Then,

$$P^2 + 2bP = 3b^2$$

$$P^2 + 2bP + b^2 = 4b^2$$

$$P + b = \pm 2b$$

$$P = b, \text{ or } -3b;$$

whence,

$$(x + a)^{\frac{1}{4}} = b, \text{ or } -3b$$

$$x + a = b^4, \text{ or } 81b^4$$

$$x = (b^4 - a), \text{ or } (81b^4 - a).$$

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## FORMATION OF EQUATIONS OF THE SECOND DEGREE.

(**225**, page 241.)

If $x = 5$; and again, $x = 3$; then, $x - 5 = 0$; (1) and $x - 3 = 0$ (2).

If we multiply $(x - 5) = 0$ by $(x - 3) = 0$, we shall have

$$x^2 - 8x + 15 = 0;$$

or,

$$x^2 - 8x = -15,$$

a common quadratic equation. By this process equations are conceived to be formed.

2. Here we suppose $x = 10$;

and $x = -7$.

Then, $(x - 10) \times (x - 7) = x^2 - 3x - 70 = 0$ is the result sought. In this manner all the results, from 2 to 8, on page 241, can be found.

By a similar process a quadratic can be resolved into two *binomial factors.*

## PROBLEMS
### PRODUCING QUADRATIC EQUATIONS.
( **230**, page 245. )

1. Let $x =$ the number; then, $4x^2 - 2x = 30$. The value of $x$ in this equation is demanded.

$$16(4x^2) - (\ ) + 4 = 30 \times 16 + 4 = 484;$$
$$4 \times 2x - 2 = \pm 22;\ x = 3,\ Ans.$$

2. Let $x =$ the number of horses;

then, $\frac{240}{x} =$ the cost of one.

If the number of horses were $x+3$, then, $\frac{240}{x+3}$ would equal the cost of one;

whence, $$\frac{240}{x} = \frac{240}{x+3} + 4.$$

Divide by 4, then multiply by $x$, and we shall have,

$$60 = \frac{60x}{x+3} + x;$$
$$\cancel{60}x + 180 = \cancel{60}x + x^2 + 3x;$$
or, $$x^2 + 3x = 180;$$
$$4x^2 + (\ ) + 9 \ = 729;$$
$$2x + 3 \quad \pm 27;\ 2x = 24;\ x = 12$$

We omit the negative root, as that *cannot represent* horses, or any positive or material thing.

3. Let $x =$ the number of sheep;

then, $\frac{240.00}{x} =$ the cost of one in cents.

He sold $x - 15$ sheep for 21600 cents;

whence. $\frac{21600}{x-15} =$ the price sold for in cents.

By conditions, $$\frac{24000}{x} + 40 = \frac{21600}{x-15}.$$

Divide by 100, and $$\frac{240}{x} + \frac{4}{10} = \frac{216}{x-15}.$$

This equation is the 13th on page 235 of the text-book, and it is reduced on a previous page in this Key.

4. Let $x =$ the number of persons;

then, $\frac{350}{x} =$ the number of cents each must pay.

If paid by $(x - 2)$ persons, $\frac{350}{x-2}$ is the sum paid by each.

By conditions, $$\frac{350}{x} + 20 = \frac{350}{x-2};$$

or, $$35 + 2x = \frac{35x}{x-2};$$

$$\cancel{35}x + 2x^2 - 70 - 4x = \cancel{35}x;\ x^2 - 2x = 35;$$

$$x - 1 = 6;\ \text{and } x = 7, \textit{Ans.}$$

5. Let $x =$ the number.

Then, $$(22 - x)x = 117$$

$$x^2 - 22x = -117$$

$$x^2 - (\quad) + 121 = 121 - 117 = 4$$

$$x - 11 = \pm 2;\ x = 13, \text{ or } 9.$$

6. Let $x =$ his rate per hour.

Then, $\frac{36}{x} =$ the number of hours on the road.

If $x$ become $x + 1$, then, $$\frac{36}{x+1} + 3 = \frac{36}{x}$$

Divide by 3, and multiply by $x$, $$\frac{12x}{x+1} + x = 12$$

$$12x + x^2 + x = 12x + 12$$

$$x^2 + x = 12;$$

whence, $$x = 3.$$

(245, 246)

Again. Let $t =$ the time, or hours, on the road.

Then, $\frac{36}{t} =$ his rate per hour.

If he were 3 hours less on the road, his rate would have been 1 greater.

Then, $\frac{36}{t-3} = \frac{36}{t} + 1$;

whence, $\cancel{36}t = \cancel{36}t - 108 + t^2 - 3t$;

or, $t^2 - 3t = 108$

$$4t^2 - (\ ) + 9 = 432 + 9 = 441$$

$$2t - 3 = 21, \text{ and } t = 12;$$

whence, $\frac{36}{12} = 3$, rate.

7. Let $x^2 =$ the gentleman's money.

Then, $\frac{1}{2}x^2 - x = 180$

$$x^2 - 2x = 360$$

$$x^2 - (\ ) + 1 = 361$$

$$x - 1 = 19,\ x = 20.$$

$$x^2 = 400, \text{ his money.}$$

8. Let $x^2 - 3 =$ the required number.

Then, $x^2 =$ sum. Square root, $x$;

whence, $x^2 - 3 + x = 17$

$$x^2 + x = 20;\ x = 4;$$

$x^2 = 16$; and $x^2 - 3 = 13$, *Ans.*

9. Let $x =$ the number sought.

Then, $x^2 + 11x = 80$

$$4x^2 + (\ ) + 121 = 320 + 121 = 441$$

$2x + 11 = \pm 21$; and $x = 5$, or $-16$.

10. Let $2x^2 =$ the number in the 1st flock.

Then, $6x + 6 =$ " " 2d flock.

$3(2x^2 + 6x + 6) =$ " " 3d flock.

$(2x^2 + 6x + 6)^2 + 6 =$ " " 4th flock.

$$\text{Sum is} \quad P^2 + 4P + 6 = 1938$$
$$\underline{\quad 2 \qquad 2}$$
$$P^2 + 4P + 4 = 1936$$
$$P + 2 = 44$$
$$2x^2 + 6x + 6 = P;$$
$$\text{whence,} \quad x^2 + 3x + 3 = 21$$
$$x^2 + 3x = 18$$
$$4x^2 + (\ ) + 9 = 9 \times 9$$
$$2x + 3 = 9, \text{ and } x = 3;$$
$$\text{whence,} \quad 2x^2 = 18, \&c.$$

11. Let $x$ represent the width of the frame.

The inside periphery of the frame is 60 inches.

The four corners of the frame are represented by $4x^2$.

Whence, $4x^2 + 60x =$ the area of the frame; but this area is to equal that of the mirror, which is $18 \times 12$.

$$\text{Whence,} \quad 4x^2 + 60x = 18 \times 12$$
$$x^2 + 15x = 18 \times 3 = 54$$

Here $a = 1$, $b = 15$, $c = 54$, $4ac = 216$, $b^2 = 225$.

$$\sqrt{4ac + b^2} = \sqrt{441} = \pm 21$$
$$x = \frac{-15 \pm 21}{2} = 3, \text{ or } -18.$$

But the width cannot be —, therefore, 3 is the *Ans.*

12. Let $x =$ the width of the walk;

then, $6x - 2 =$ a side of the court;

and $4x(6x - 2) + 4x^2 =$ the sq. yds. in the walk, or its area.

$4(6x - 2) =$ the yards in periphery of court-yard.

By a condition given,

$$4x(6x-2)+4x^2=4(6x-2)+164.$$

Dividing by 4, and

$$6x^2-2x+x^2=6x-2+41$$

$$7x^2-8x=39$$

$$4\times 7^2x^2-(\ )+64=39\times 28+64=1156.$$

Square root, $\quad 2\times 7x-8=\pm 34;$

$$7x-4=\pm 17;\ \text{and } x=3, \text{ or } \tfrac{13}{7}.$$

The negative roots do not apply to material things, as we have before explained. Therefore, we must take 3 for the only true value of $x$. The answer is the value of

$$4x(6x-2)+4x^2;$$

whence, $\quad x=3$; that is, $228=256$, *Ans.*

13. Let $\quad x=$ B's rate of travel per hour;

then, $\quad x+3=$ A's rate;

whence, $\quad \dfrac{150}{x}=$ B's time;

and $\quad \dfrac{150}{x+3}=$ A's time.

But their difference in time is $8\frac{1}{3}$; therefore,

$$\frac{150}{x}-\frac{150}{x+3}=\frac{25}{3}.$$

Dividing by 25, and, $\dfrac{6}{x}-\dfrac{6}{x+3}=\dfrac{1}{3}.$

For the sake of variety, let $a=3$; then this last equation becomes

$$\frac{2a}{x}-\frac{2a}{x+a}=\frac{1}{a};$$

$$2a^2-\frac{2a^2x}{x+a}=x;$$

$$2a^2x+2a^3-2a^2x=x^2+ax;$$

or, $$x^2+ax=2a^3;$$

$$4x^2+(\ )+a^2=8a^3+a^2=a^2(8a+1);$$

$$=25a^2;$$

Square root, $\quad 2x+a=5a;\ 2x=4a;$

whence, $\quad x=2a=6$, and $x+3=9$, *Ans.*

14. Let $x =$ the number of persons;

then, each must pay $\frac{175}{x}$;

but, after two had left, each must pay $\frac{175}{x-2}$, and the difference is 10 cents.

Then, $$\frac{175}{x} + 10 = \frac{175}{x-2}.$$

Dividing by 5, and multiplying by $x$, produces,

$$35 + 2x = \frac{35x}{x-2};$$

$$35x - 70 + 2x^2 - 4x = 35x;$$

whence, $$x^2 - 2x = 35;\ x - 1 = 6;\ x = 7.$$

15. Let $19x =$ the whole distance;

then, $x =$ B's rate per day, also the days he traveled;

and $x^2 =$ B's distance.

A had passed over, during this time, $x$ days, $7x$ miles;

Hence, $x^2 + 7x + 32 = 19x$; $x^2 - 12x = -32$;

$$x^2 - 12x + 36 = 4;$$

$$x - 6 = \pm 2$$

$$x = 8, \text{ or } 4;$$

whence, $19 \times 4 = 76$, or $19 \times 8 = 152$, the whole distance.

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## QUADRATIC EQUATIONS,

### CONTAINING TWO UNKNOWN QUANTITIES.

(**232**, page 251.)

1. Double the 1st equation, and add to the 2d;

then, $x^2 + 4x = 77$; $x^2 + 4x + 4 = 81$;

$$x + 2 = \pm 9.$$

These values of $x$, substituted in either of the given equations, will give the value of $y$.

2. Obtain two values of $x^2$, and equate them; then, we shall have, $(3y + 1)^2 = 3y^2 + 13$;

or, $$6y^2 + 6y = 12;$$

$$y^2 + y = 2; \quad y = 1, \text{ or } -2;$$

and, $$x = 3y + 1 = 4, \text{ or } -5.$$

3. Multiply the 1st equation by $x$, from the product,

$$x^2 + xy = 12x,$$

subtract, $$xy = 35,$$

and, $$x^2 - 12x = -35;$$

$$x^2 - 12x + 36 = 1;$$

$$x - 6 = \pm 1; \text{ and } x = 7, \text{ or } 5.$$

4. Multiply the 1st equation by $x$, and $x^2 - xy = -x$; that is, $x^2 + x = 42$; whence, $x = 6$, or $-7$.

5. Divide the 2d by the 1st, and

$$\left.\begin{array}{l} x - y \quad 1; \\ \text{but, } x + y = 1125; \end{array}\right\} \begin{array}{l} \text{whence, } 2x = 1126, \\ x = 563, \text{ and } y = 562. \end{array}$$

6. Divide the 2d by the 1st, and

$$x^2 + xy + y^2 = 31;$$

but, $x^2 - 2xy + y^2 = 16$, the square of the 1st

By subtraction, $$3xy = 15;$$

$$xy = 5$$

Add, $$x^2 + xy + y^2 = 31$$

Sum, $$x^2 + 2xy + y^2 = 36$$

Square root, $$x + y = 6, \text{ or } -6;$$

But, $$x - y = 4,$$

Sum, $$2x = 10, \text{ and } x = 5, \text{ or } -1;$$

whence, $$y = 1, \text{ or } -5.$$

7. Divide first equation by $x+y$.

Then, $x^2 - xy + y^2 = 19$

Squaring 2d, and $x^2 - 2xy + y^2 = 9$

By subtraction, $xy = 10$

$3xy = 30$

Add $x^2 - xy + y^2 = 19$

Sum $x^2 + 2xy + y^2 = 49$

Square root, $x + y = 7$, or $-7$

But $x - y = 3$

Sum, $2x = 10$, or $-4$

$x = 5$, or $-2$; whence, $y=2$, or $=-5$.

8. Put $P = \frac{1}{x}$, and $Q = \frac{1}{y}$, as suggested in the text-book, and the equations become

$$P + Q = \frac{5}{6} \qquad (1)$$

and

$$P^2 + Q^2 = \frac{13}{36} \qquad (2)$$

Squaring (1), and subtracting (2), there will be left,

$$2PQ = \frac{12}{36} \qquad (3)$$

From (2) subtract (3), and

$$P^2 - 2PQ + Q^2 = \frac{1}{36}.$$

Square root, $P - Q = \frac{1}{6}$, or $-\frac{1}{6}$ (4)

But, (1), $P + Q = \frac{5}{6}$,

Sum, $2P = 1$, or $\frac{4}{6}$

$P = \frac{1}{2}$, or $\frac{1}{3}$

Subtract (4) from (1), and

$$2Q = \frac{4}{6}, \text{ or } 1;$$

$$Q = \frac{1}{3}, \text{ or } \frac{1}{2};$$

whence, $$\frac{1}{x} = \frac{1}{2}, \text{ or } \frac{1}{3};$$

$$x = 2, \text{ or } 3; \quad y = 3, \text{ or } 2.$$

9. Divide the 1st by the 2d.

Then, $$x^2 - xy + y^2 = 19 \qquad (1)$$

Square of the 2d, $$x^2 + 2xy + y^2 = 64 \qquad (2)$$

Subtract (1) from (2), and $$3xy = 45$$

$$xy = 15 \qquad (3)$$

Subtract (3) from (1), and

$$x^2 - 2xy + y^2 = 4$$

Square root, $$x - y = 2, \text{ or } -2$$

But $$x + y = 8 \qquad 8$$

Sum, $$2x = 10, \text{ or } -6$$

$$x = 5, \text{ or } \quad 8$$

Diff. $$2y = 6, \text{ or } \quad 10$$

$$y = 3, \text{ or } \quad 5.$$

10. Double the 2d equation, and add it to the 1st; then,

$$x^2 + 2xy + y^2 + x + y = 156;$$

or, $$(x + y)^2 + (x + y) = 156.$$

Place $$x + y = P;$$

then, $$P^2 + P = 156$$

$$4P^2 + 4P + 1 = 625$$

$$2P + 1 = \pm 25$$

$$2P = 24, \text{ or } -26,$$

$$P = 12, \text{ or } -13;$$

that is, $x + y = 12$, or $-13$.

But, $xy + x + y = 39$;

whence, $xy = 27$, or $52$.

But, $x + y = 12$, or $-13$.

Multiply by $x$, $x^2 + xy = 12x$, or $-13x$;

that is, $x^2 + 27 = 12x$; or, $x^2 + 27 = -13x$;

or, $x^2 + 52 = 12x$; or, $x^2 + 52 = -13x$.

Here are four distinct quadratics. Each one has two roots, or two values for $x$. The whole number of values for $x$ will therefore be 8.

Some of them may be imaginary, or impossible. Such are called, in higher algebra, *roots of solution.*

The 8 roots are,

Real roots, $\begin{cases} x = 9 \text{ or } 3 \\ x = -6\frac{1}{2} + \frac{1}{2}\sqrt{61}, \text{ or } -6\frac{1}{2} - \frac{1}{2}\sqrt{61} \end{cases}$

Imaginary roots, $\begin{cases} x = 6 + 4\sqrt{-1}, \text{ or } 6 - 4\sqrt{-1} \\ x = 6\frac{1}{2} + \sqrt{-39}, \text{ or } -6\frac{1}{2} - \sqrt{-39} \end{cases}$

There must be 8 corresponding values of $y$. The real and practical roots only are given in the text-book.

11. Because the 2d members are the same in each equation, we have

$$2x^2 - 3xy = x^2 - 2y^2;$$

whence, $x^2 - 3xy = -2y^2$.

Conceive $3y$ to be a known coefficient of $x$, and complete the square; then,

$$x^2 - 3y \times x + \frac{9y^2}{4} = \frac{9y^2}{4} - 2y^2 = \frac{y^2}{4};$$

extract square root, $x - \frac{3}{2}y = \pm\frac{y}{2}$;

$$x = 2y, \text{ or } y;$$

whence, $x^2 = 4y^2$, or $y^2$.

Substitute these values of $x^2$ in the 2d of the given equations, and $4y^2 - 2y^2$, or $2y^2 = 50$; $y^2 = 25$, or $y = \pm 5$;
or, $y^2 - 2y^2$, or $-y^2 = 50$; $y^2 = -50$, or $y = \pm 5\sqrt{-2}$

The value of $x$ is double that of $y$. Hence, $x = 10$.

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## PROBLEMS

### PRODUCING QUADRATICS OF TWO UNKNOWN QUANTITIES.

( **233**, page 252. )

1. Let $x^2 + y^2 = 100$ (1) Then, $x + y = 14$ (2)

Sq. (2), $\underline{x^2 + 2xy + y^2 = 100 + 80 + 16}$

Diff. $2xy = 96$ (3)

Subtract (3) from (1), and

$$x^2 - 2xy + y^2 = 4$$
$$x - y = 2, \text{ or } -2$$
$$(2) \quad \underline{x + y = 14}$$
$$2x = 16, \text{ or } 12;\ x = 8, \text{ or } 6$$

Whence, 64 and 36, *Ans.*

2. This is the same as Ex. 1, in other words.

3. This is the same as Ex. 1, generalized.

That is, $x + y = a$ (1)

$x^2 + y^2 = b$ (2)

Solved the same as Ex. 1.

4. Let $x$ and $y$ = the numbers; then,

$$x + y = 24; \quad (1)$$

and $$xy = 35(x - y). \quad (2)$$

Let $x = vy$; then, (1) becomes $vy + y = 24$; (3)

and $$vy^2 = 35(vy - y). \quad (4)$$

From (3), $y = \dfrac{24}{v+1}$; from $= \dfrac{35(v-1)}{v}$.

Equating values of $y$,

and $24v = 35v^2 - 35$

$$35v^2 - 24v = 35$$

$$v^2 - \frac{24v}{35} = 1$$

$$v^2 - (\ ) + \left(\frac{12}{35}\right)^2 = \frac{144}{(35)^2} + \frac{(35)^2}{(35)^2} = \frac{1369}{(35)^2}$$

$$v - \frac{12}{35} = \pm\frac{37}{35};\ v = \frac{49}{35} = \frac{7}{5},\ \text{or} -\frac{5}{7};$$

whence, $y = \dfrac{24}{\frac{7}{5}+1} = \dfrac{24 \times 5}{12} = 10;$

or, $y = \dfrac{24}{-\frac{5}{7}+1} = \dfrac{24 \times 7}{2} = 84.$

The values of $y$, substituted in $x + y = 24$,

give $x = 14$, or $x = -60$.

That is, the numbers may be 84 and $-60$; sum $= 24$;

product, $84 \times 60 = 35 \times 144$.

As these products are equal, these numbers answer the condition of the problem.

5. This is the same as Example 9, on page 251, text-book

6. Let $x =$ the greater number, and $y$ the less.

Then, $y : x :: x : 12;$ (1)

and $x^2 + y^2 = 45.$ (2)

From (1), $x^2 = 12y.$

This, substituted in (2), will give,

$$y^2 + 12y = 45$$

$$y^2 + (\ ) + 36 = 81$$

$$y + 6 = \pm 9;\ y = 3,\ \text{or} -15;$$

whence, $x^2 = 12 \times 3$, or $-12 \times 15$

That is, $x = \pm 6$, or $6\sqrt{-5}$.

7. This is solved the same as Example 6, on page 251 of the text-book.

8. This is solved the same as Example 9, on page 251 of the text-book.

9. Let $x =$ the yards of silk, and $y =$ the yards of cloth;

then, $x + y = 110,$ (1)

and, $80y - x^2 = 400.$ (2)

From (1), $80y = 80(110) - 80x.$ (3)

Equating the two values of $80y$, and we obtain,

$$400 + x^2 = 80(110) - 80x$$

$$x^2 + 80x = 8800 - 400 = 8400$$

$$x^2 + 80x + 1600 = 10000$$

$$x + 40 = \pm 100;\ x = 60,$$

But, $x + y = 110$; whence, $y = 50$, *Ans.*

10. Let $x - 2 =$ the age of B; then, $x + 2 =$ the age of A.

$x^2 - 4x + 4 =$ square of B's age,

$x^2 + 4x + 4 =$ " A's age.

Sum, $2x^2 + 8 = 976;$

$$x^2 = 484 = 4(121)$$

$$x = 2 \times 11 = 22$$

$x - 2 = 20$, B's age,

and $x + 2 = 24$, A's age, *Ans.*

11. Let $x =$ the greater part, and $y =$ the less;

then, $x + y = 10$ (1)

$$(4y)^2 - 112 = (2x)^2 \quad (2)$$

$$4y^2 - 28 = x^2 \quad (3)$$

From (1), $(10 - y)^2 = x^2;$ (4)

that is, $4y^2 - 28 = 100 - 20y + y^2;$

$$3y^2 + 20y = 128.$$

Here, $a = 3$, $b = 20$, and $c = 128$; $4ac = 128 \times 12$;

$$\sqrt{4ac + b^2} = \sqrt{1936} = \pm 44;$$

$$y = \frac{-20 \pm 44}{6} = 4, \text{ less part.}$$

12. Let $x$ and $y$ represent the numbers;

then, $x^2 + y^2 = 89$ (1)

$(x + y)x = 104$ (2)

Let $x = vy$; then, (1) becomes,

$$v^2y^2 + y^2 = 89; \; y^2 = \frac{89}{v^2 + 1};$$

and (2) becomes, $v^2y^2 + vy^2 = 104$; $y^2 = \dfrac{104}{v^2 + v}$.

Equating these two values of $y^2$, we obtain,

$$\frac{104}{v^2 + v} = \frac{89}{v^2 + 1}$$

$$104v^2 + 104 = 89v^2 + 89v$$

$$15v^2 - 89v = -104$$

$$4 \times (15)^2v^2 - (\;) + \overline{89}^2 = 7921 - 6240 = 1681$$

$$2 \times 15v - 89 = \pm 41$$

$$30v = 130, \text{ or } 48; \; v = \tfrac{13}{3}, \text{ or } \tfrac{16}{10} = \tfrac{8}{5};$$

but, $y^2 = \dfrac{89}{v^2 + 1} = \dfrac{89}{\frac{64}{25} + 1} = \dfrac{89 \times 25}{89} = 25;$

therefore, $y = \pm 5,$

If $y = 5$, and $v = \frac{8}{5}$, and $x = vy$, then, $x = 8,$ } *Ans.*

13. Let $x =$ the digit in the place of tens, and $y =$ the digit in the place of units. Then $10x + y =$ the number sought.

$$\frac{10x + y}{x + y} = \frac{32}{5} \text{ by one condition.} \quad (1)$$

And $10x + y - 9 = 10y + x$ (2)

From (1) $50x + 5y = 32x + 32y$;

or, $18x = 27y$; or, $2x = 3y$. (3)

From (2) $9x - 9 = 9y$;

or, $x - 1 = y$; $2x - 2 = 2y$, (4)

subtract (4) from (3), and $2 = y$.

This value of $y$, substituted in 3, and $x = 3$; whence, 32 is the number sought.

14. Let $x =$ the 1st, and least number;

$y =$ the 2d, or next greater number;

and $P =$ the 3d, or greatest number.

Then,

$$x + y + P = 20 \qquad (1)$$

$$xyP = 270 \qquad (2)$$

$$y - x + 2 = P - y \qquad (3)$$

Add (1) and (3), and $2y + P + 2 = 20 + P - y$;

whence $3y = 18$, and $y = 3$.

This value of $y$, substituted in (1) and (2), and we obtain,

$$x + P = 17$$

$$xP = 90.$$

Now $x$ and P can be discovered by the same process that $x$ and $y$ were discovered in Example 3, page 251, text-book, the solution being in this Key.

15. Let $x - 2 =$ the number in a side of one square;

and $x + 2 =$ the number in a side of the other.

Then,

$$x^2 - 4x + 4$$

$$x^2 + 4x + 4$$

$$2x^2 + 8 = 1066$$

$$x^2 = 529,\ x = 23,$$

$x - 2 = 21$, and $x + 2 = 25$, *Ans.*

16. Let $x$ = the number of bushels of wheat.

Then, $x + 16$ = the number of bushels of barley;

$\frac{24}{x}$ = the price per bushel for wheat;

and $\frac{24}{x+16}$ = the price per bushel for barley.

$$\text{Then,} \quad \frac{24}{x} = \frac{24}{x+16} + \frac{25}{100} \qquad \frac{25}{100} = \frac{1}{4}$$

$$\frac{96}{x} = \frac{96}{x+16} + 1$$

$$96x + 16 \times 96 = 96x + x^2 + 16x. \quad \text{Let } a = 8.$$

$$\text{Then,} \quad x^2 + 2ax = 96 \times 2a$$

$$x^2 + 2ax = 12a \times 2a = 24a^2$$

$$x^2 + (\quad) + a^2 = 25a^2$$

$$x + a = \pm 5a, \ x = 4a = 32.$$

17. Let $x$ = the length of one trench,

and $x + 6$ = the length of the other;

whence, $x^2 + (x+6)^2 = 356$. 17£. 16s. = 356s.

$$2x^2 + 12x + 36 = 356$$

$$x^2 + 6x + 18 = 178$$

$$\text{Subtract} \qquad 9 = 9$$

$$x^2 + 6x + 9 = 169$$

$x + 3 = 13$, $x = 10$, and $x + 6 = 16$, *Ans.*

18. Let $x$ = B's rate per day;

$9$ = A's " "

$x + 3$ = the days each traveled.

$$\text{Then,} \quad (x+3)(x+9) = 247$$

$$x^2 + 12x + 27 = 247$$

$$9 = 9$$

$$x^2 + 12x + 36 = 256$$

$$x + 6 = 16, \text{ and } x = 10,$$
$$x + 3 = 13 \text{ days; whence}$$
$$\left.\begin{array}{l} 13 \times \ \ 9 = 117, \text{ A's distance;} \\ 13 \times 10 = 130, \text{ B's} \quad \text{"} \end{array}\right\} \textit{Ans.}$$

19. Let $x =$ the circumference of the hind wheels;
and $y =$ " " fore "

Then, $\frac{120}{x} =$ the number of revolutions of the hind wheels;

and $\frac{120}{y} =$ " " " fore wheels;

By 1st condition, $\frac{120}{x} + 6 = \frac{120}{y};$ (1)

By 2d condition, $\frac{120}{x+1} + 4 = \frac{120}{y+1}.$ (2)

From (1), $20y + xy = 20x;$ (3)

whence, $y = \frac{20x}{20+x}.$ (4)

From (2), $\frac{30}{x+1} + 1 = \frac{30}{y+1};$

or, $30y+30+xy+x+y+1 = 30x + 30,$
$$31y + xy + 1 = 29x; \quad (5)$$

whence, $y = \frac{29x-1}{31+x}.$ (6)

Equating (4) and (6), and
$$\frac{20x}{20+x} = \frac{29x-1}{31+x}$$
$$620x + 20x^2 = 580x + 29x^2 - 20 - x$$
$$41x + 20 = 9x^2$$
$$9x^2 - 41x = 20$$
$$4 \times 9^2x^2 - (\ ) + \overline{41}^2 = 36 \times 20 + \overline{41}^2 = 2401$$

Square root, $2 \times 9x - 41 = \pm 49$

$$\left.\begin{array}{r} 2 \times 9x = 90; \quad x = 5, \\ \text{This value placed in (4), and } y = \frac{20 \times 5}{25} = 4, \end{array}\right\} \textit{Ans.}$$

20. Let $x$ and $y$ represent the numbers.

Then, $$xy = 120 \qquad (1)$$

$$(x + 2)(y - 3) = 120 \qquad (2)$$

or, $$xy + 2y - 3x - 6 = 120 = xy \qquad (3)$$

$$2y = 3x + 6$$

$$2xy = 3x^2 + 6x = 2 \times 120$$

$$x^2 + 2x = 2 \times 40 = 80$$

$$x^2 + 2x + 1 = 81$$

$$x + 1 = \pm 9; \ x = 8.$$

This value of $x$, placed in (1), and

$$8y = 120; \ 2y = 30; \ y = 15.$$

21. Let $x$ and $y$ represent the numbers in question.

Then, $$x^2 + y^2 = 2xy + 4; \qquad (1)$$

$$x^2 - y^2 = \tfrac{1}{2}xy + 4. \qquad (2)$$

By subtraction, $$2y^2 = \tfrac{3}{2}xy;$$

$$y = \tfrac{3}{4}x$$

From (1), $x^2 - 2xy + y^2 = 4.$

Square root, $x - y = 2;$

that is, $x - \tfrac{3}{4}x = 2;$ or, $x = 8;$

whence, $y = 6.$

22. Let $x$ and $y$ represent the numbers in question.

Then $$\sqrt{27x} = 27y; \text{ and } \sqrt[3]{3x} = 3y.$$

$$27x = 27 \times 27y^2; \text{ and } 3x = 3^3y^3;$$

$$x = 27y^2; \qquad x = 9y^3;$$

whence, $$9y^3 = 27y^2;$$

$$y = 3,$$

$$x = 9 \times 27 = 243.$$

23. Let $x =$ the cost of the horse;
and, $x - 24 =$ the sum lost.
Then, $x : x - 24 :: 100 : x;$

$$x^2 - 100x = -2400$$
$$x^2 - 100x + \overline{50}^2 = 2500 - 2400 = 100$$
$$x - 50 = \pm 10$$
$$x = 40, \text{ or } 60, \textit{ Ans.}$$

24. Let $x$ and $y$ represent the numbers;
then, $xy = x^2 - y^2;$ (1)
and $x : y :: 3 : 2$

$$2x = 3y. \quad (2)$$

Double (1), and we have, $2xy = 2x^2 - 2y^2.$ (3)
The value of $2x$, found in (2), substituted in (3), will give,

$$3y^2 = 2x^2 - 2y^2; \text{ whence, } 2x^2 = 5y^2.$$

From (2), $2x^2 = \frac{9y^2}{2};$

whence, $5y^2 = \frac{9y^2}{2}$, or $10y^2 = 9y^2$, which is absurd.
Therefore, the problem is an impossible one.

25. Let $x$ and $y$ represent the numbers,
then, $2xy + 9 = x^2 + y^2,$ (1)
and $\frac{1}{2}xy + 9 = x^2 - y^2.$ (2)
From (1), $x^2 - 2xy + y^2 = 9.$
Square root, $x - y = \pm 3$, or $y = x - 3$, or $x + 3$.
Add (1) and (2), and $\frac{5}{2}xy + 18 = 2x^2,$

$$5xy + 36 = 4x^2;$$

that is, $5x(x - 3) + 36 = 4x^2,$
or, $x^2 - 15x + 36 = 0;$ whence, $x = 12.$

## ARITHMETICAL PROGRESSION.

There are two fundamental equations which, together, embrace the whole subject; they are as follows:

$$L = a \pm (n - 1)d, \quad \text{(A)}$$
$$S = \tfrac{1}{2}(a + L)n. \quad \text{(B)}$$

( **238**, page 258. )

1. Here, $a = 5$, $L = 92$, $n = 30$. S is required, and equation (B) will give it. Thus,
$S = \frac{1}{2}(5 + 92)30 = \frac{1}{2} \times 97 \times 30 = \frac{1}{2}(2910) = 1455$, *Ans.*

2. Here, $a = 2$, $n = 10$, $L = 30$.
$$S = (2 + 30)5 = 160.$$

3. Here, $a = 5$, $L = 107$, $n = 35$.
$$S = (5 + 107)\tfrac{35}{2} = 56 \times 45 = 1960.$$

4. Here, $a = 7$, $L = 207$, $n = 21$.
$$S = \tfrac{1}{2}(214)21 = 107 \times 21 = 2247.$$

5. Here, $a = 6$, $L = -3\frac{1}{2}$, $n = 20$.
$$S = (6 - 3\tfrac{1}{2})10 = 25.$$

PROBLEMS.

( **239**, page 260. )

3. Here, $L = 49$, $a = 1$, $n = 9$; $d$ is required.
Apply equation (A), $\quad 49 = 1 + 8d$;
whence, $\quad 8d = 48$, $d = 6$.
Then, the required numbers must be 1, 7, 13, 19, 25, &c.

4. Here $a = 1$, $S = 280$, $n = 32$. $d$ and L required.

$$L = 1 + 31d; \qquad \text{(A)}$$

$$280 = (a + 1 + 31d)16. \qquad \text{(B)}$$

Dividing (B) by 8, $35 = (1 + 1 + 31d)2 = 4 + 62d$.

$$62d = 31; \; d = \tfrac{1}{2},$$

This value of $d$, substituted in (A), gives

$$L = \frac{2}{2} + \frac{31}{2} = 16\tfrac{1}{2}, \textit{ Ans.}$$

5. Here $a = \frac{1}{3}$, $L = \frac{1}{2}$, $n = 5$. $d$ and S required.

$$S = \left(\frac{1}{3} + \frac{1}{2}\right)\frac{5}{2} = \frac{5}{6} \times \frac{5}{2} = \frac{25}{12}$$

$$\frac{1}{2} = \frac{1}{3} \pm 4d;$$

or, $$3 = 2 + 24d; \; d = \frac{1}{24}.$$

$$\frac{1}{3} + \frac{1}{24} = \frac{9}{24} = \frac{3}{8}, \text{ the first mean.}$$

$$\frac{3}{8} + \frac{1}{24} = \frac{10}{24} = \frac{5}{12}, \text{ the second mean.}$$

6. Here $a = 5$, $L = 15$, $n = 7$. $d$ and L required.

$$15 = 5 + 6d; \; d = \frac{5}{3} = 1\tfrac{2}{3}.$$

Hence, $5 + 1\frac{2}{3} = 6\frac{2}{3}$, the first mean.

7. To put the first ball into the basket, the man must travel 2 yards, and to put the last ball in he must travel 200 yards. Hence, $a = 2$, $L = 200$, $d = 2$, $n = 100$.

$$S = (2 + 200)\frac{100}{2} = 10100 \text{ yards} = 5 \text{ miles } 1300 \text{ yards.}$$

8. Here $a = 10$, $d = 20$, $n = 47$. L and S required.

$$L = 10 + 46 \times 20 = 930;$$

$$S = (10 + 930)\frac{47}{2} = 470 \times 47 = 22090 \text{ dollars.}$$

9. Here $a = 4$, $d = 4$, $S = 20200$. $n$ is required.

$$20200 = (4 + L)\frac{n}{2};$$

$$L = 4 + (n - 1)4 = 4n;$$

$$20200 = (4 + 4n)\frac{n}{2} =$$

$$n^2 + n = 10100;$$

$$4n^2 + 4n + 1 = 40401;$$

$$2n + 1 = 201; \; n = 100, \; \textit{Ans.}$$

10. $S = (1 + 24)12 = 25 \times 12 = 300.$

11. Here $a = 730$, $d = 2$, $L = 2$. $n$ is required.

$$2 = 730 \pm (n - 1)2;$$

$$(n - 1)2 = 728; \; n = 365, \; \textit{Ans.}$$

12. Here $S = 280$, $a = 1$, $n = 32$. $d$ is required.

$$280 = (1 + L)16; \qquad \text{(B)}$$

$$L = 1 + (n - 1)d. \qquad \text{(A)}$$

Reducing (B), we find $35 = 2 + 2L$; $L = \frac{33}{2}$

This value of L, substituted in (A), will give

$$\frac{33}{2} = 1 + 31d; \text{ whence, } d = \frac{1}{2}.$$

13. Here $S = 950$, $d = 3$, $n = 25$. $a$ is required.

$$L = a + 24 \times 3;$$

$$950 = (a + 24 \times 3)\frac{25}{2};$$

$$950 = (a + 36)25;$$

$$38 = a + 36; \; a = 2, \; \textit{Ans.}$$

14. Here $a = 1$, $L = n$, and $n = n$;

$$S = (1 + n)\frac{n}{2}. \qquad \text{(B)}$$

(260, 261)

(**240**, page 263.)

2. Here the numbers may be represented by $x - 3y$, $x - y$, $x + y$, and $x + 3y$.

Then, $(x - y) + (x + y) = 2x = 25$; (1)

and $(x - y)\,2y = 50.$ (2)

$$2xy - 2y^2 = 50\text{; but } 2xy = 25y\text{, from (1)}$$

$$2y^2 - 25y = -50$$

$$16y^2 - (\ ) + \overline{25}^2 = 625 - 400 = 225$$

$$4y - 25 = \pm 15 \qquad y = \tfrac{10}{4} = 2\tfrac{1}{2}$$

$$x = 12\tfrac{1}{2};$$

whence, $x - y = 10$, and $x + y = 15$, the means.

3. Take the same notation as before.

Then, by the conditions we obtain

$$(x - 3y)(x + y) = 5 \qquad (1)$$

$$(x + 3y)(x - y) = 21 \qquad (2)$$

From (1), $x^2 - 2xy - 3y^2 = 5$ (3)

" (2), $x^2 + 2xy - 3y^2 = 21$ (4)

By subtraction, $4xy = 16$;

or, $xy = 4$; $y^2 = \dfrac{16}{x^2}$

By addition, $2x^2 - 6y^2 = 26$

$$x^2 - 3y^2 = 13$$

$$x3 - \frac{3 \times 16}{x^2} = 13$$

$$x^4 - 13x^2 = 48;$$

whence, $x = 4$;

then, $y = 1$;

and $x - y = 3$, $x + y = 5$, the means.

5. Let $x-y$, $x$ and $(x+y)$, represent the numbers.

Then, $3x = 15$; or, $x = 5$;

and $(x^2 - y^2)\,x = 105$;

whence, $y = 2$.

6. Here, $3x = 18$ and $x = 6$;

whence, $3x^2 + 2y^2 = 158$;

and $y = 5$.

7. Here $\left.\begin{array}{l} 3x^2 + 2y^2 = 56 \\ 6x + 2y = 28 \end{array}\right\}$ by the conditions, (1) (2)

and

From (2), $y = 14 - 3x$ (3)

$$y^2 = 196 - 84x + 9x^2$$

$$2y^2 = 392 - 168x + 18x^2.$$

This value of $2y^2$, placed in (1), produces

$$21x^2 - 168x + 392 = 56$$

$$21x^2 - 168x + 336 = 0$$

$$x^2 - 8x + 16 = 0$$

$$x - 4 = 0; \text{ or, } x = 4, \text{ the mean.}$$

8. Let $x-y$, $x$, and $x+y$, represent the numbers.

Then, $3x = 12$; or, $x = 4$; and $x^3 = 64$

$$(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

$$x^4$$

Sum, $3x^4 + 12x^2y^2 + 2y^4 = 962$

Dividing by 2, $\frac{3x^4}{2} + 6x^2y^2 + y^4 = 481.$

$$\frac{3x^4}{2} = \frac{3 \times 256}{2} = 384.$$

This taken from each member,

and $y^4 + 6x^2y^2 = 97$

$$y^4 + 6x^2y^2 + 9x^4 = 2304 + 97 = 2401$$

$$y^2 + 3x^2 = \pm 49 \qquad y^2 + 48 = 49$$

Whence, $y^2 = 1$; and $y = 1$.

9. Here, $(x-y)^2+(x+y)x=28,$ (1)

$(x+y)^2+(x-y)x=44.$ (2)

From (1), $2x^2-xy+y^2=28;$

" (2), $2x^2+xy+y^2=44.$

By subtraction and division, $xy=8.$ (3)

By addition and division, $2x^2+y^2=36.$ (4)

But, from (3), $y^2=\frac{64}{x^2}$; then, $2x^2+\frac{64}{x^2}=36$

$$2x^4-36x^2+64=0$$
$$x^4-18x^2+32=0$$
$$x^4-18x^2+81=81-32=49$$
$$x^2-9=\pm 7$$

whence, $x=4$, the mean.

10. Here the mean is 5;

$$3x^2+2y^2=93;\ 2y^2=93-75=18;$$

and $y=3.$

11. Here $2x=8$, and $x=4$, the mean.

Hence, $16+(4+y)^2=52$

$(4+y)^2=36;\ 4+y=6;$ and $y=2.$

12. Let $x-3y$, $x-y$, &c., represent the numbers.

Then, $2x=13;\ x=6\frac{1}{2};$ and $4xy=39;$

But, $4x=26;$

whence, $26y=39;\ 2y=3;$ and $y=1\frac{1}{2}.$

$x-y=5;$ and $x+y=8$, the means.

13. Let $x=$ the fourth term, and $y$ the common difference.

Then, $(x-3y), (x-2y), (x-y), x, (x+y), (x+2y), (x+3y)$, represent the seven numbers;

$x - 3y =$ the 1st number,
$x + 2y =$ the 6th.

$$2x - y = 14 \qquad (1)$$

$$(x - y)(x + y) = 60; \qquad (2)$$

or, $$x^2 - y^2 = 60.$$

But, from (1), $y^2 = (2x + 14)^2 = 4x^2 + 56x + 196.$

That is, $-3x^2 - 56x - 196 = 60$

$$3x^2 - 56x = -256$$

$$36x^2 - (\ ) + \overline{56}^2 = 56^2 - 256 \times 12 = 64$$

$$6x - 56 = \pm 8$$

$$6x = 48;$$

$$x = 8, \text{ and } y = 2, \textit{Ans.}$$

14. Let $x =$ the third number, and $y$ the common diff., and we have, $(x - 2y)$, $(x - y)$, $x$, $(x + y)$, $(x + 2y)$.

Sum, $5x = 25$, and $x = 5$, the third number;

$$5)\underline{945}$$

$$189 = \text{the product of the other 4 numbers.}$$

Prod. of 1st and 5th, $(x - 2y)(x + 2y) = x^2 - 4y^2;$
" 2d and 4th, $(x - y)\ (x + y) = x^2 - y^2.$

Continued product, $x^4 - 5x^2y^2 + 4y^4 = 189.$

But, $x = 5$, and $x^4 = 625;$

then, $625 - 125y^2 + 4y^4 = 189;$

and $436 - 125y^2 + 4y^4 = 0.$

Let $y = 2\text{P};$

then, $436 - 125 \times 4\text{P}^2 + 4 \times 16\text{P}^4 = 0.$

By dividing by 4, we obtain $109 - 125\text{P}^2 + 16\text{P}^4 = 0.$

Now, because $109 + 16 = 125$, the sum of all the coefficients must equal 0 on each side, therefore P cannot be greater nor less than 1. Hence, $\text{P} = 1$ $y = 2\text{P} = 2$. But $x = 5$; whence the numbers are 1, 3, 5, 7, 9.

15. Let $(x-3y)$, $(x-y)$, $(x+y)$, $(x+3y)$, represent the numbers.

$$\begin{array}{rl} x^2 - 2xy + \phantom{9}y^2 & \\ x^2 - 6xy + 9y^2 & \\ \hline 4xy - 8y^2 = 12 & (1) \\ 4xy + 8y^2 = 28 & (2) \\ \hline \end{array}$$

Difference, $16y = 16$; whence, $y = 1$.

From (1), $4x - 8 = 12$, $x - 2 = 3$, and $x = 5$.

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## GEOMETRICAL PROGRESSION.

( **249**, page 275. )

4. Let $x =$ the first term, and $y =$ the ratio.

Then $x$, $xy$, $xy^2$ and $xy^3$, will represent the numbers.

And $$\left.\begin{array}{rl} x + xy^2 = 20 & (1) \\ xy + xy^3 = 60 & (2) \end{array}\right\} \text{By the given conditions.}$$

Dividing (2) by (1), $y = 3$;

then (1) becomes $10x = 20$, and $x = 2$.

Hence 2, 6, 18, and 54 are the numbers sought.

5. Here the numbers are $x$, $xy$, $xy^2$.

Whence, $x + xy + xy^2 = 210$; (1)

and $xy^2 - x = 90$. (2)

By addition, $xy + 2xy^2 = 300$. (3)

By subtraction, $2x + xy = 120.$ (4)

Dividing (3) by (4), and $\frac{y + 2y^2}{2 + y} = \frac{10}{4} = \frac{5}{2}$

$$2y + 4y^2 = 10 + 5y$$

$$4y^2 - 3y = 10;$$

whence, $y = 2.$

But (2), $xy^2 - x = 90$

Therefore $3x = 90$; $x = 30$; and 30, 60, 120, *Ans.*

6. Here, $x + xy + xy^2 + xy^3 = 30$ (1)

$$\frac{xy^3}{xy + xy^2} = \frac{y^2}{1 + y} = \frac{4}{3} \qquad (2)$$

$$3y^2 - 4y = 4;$$

whence, $y = 2.$

Now, (1) becomes, $x + 2x + 4x + 8x = 30,$

therefore, $15x = 30;$

and, $x = 2.$

7. Here, using the same notation as in the last problem, we have $x + xy^2 = 148;$ (1)

and $xy + xy^3 = 888.$ (2)

Dividing (2) by (1), and we obtain $y = 6.$

From (1), $x + 36x = 148$

$$37x = 148$$

$$x = 4$$

whence, 4, 24, 144, 864, *Ans.*

8. Let $x$, $xy$, and $xy^2$, be the numbers.

Then, $$x^3y^3 = 216;$$

whence, $$xy = 6; \quad (1)$$

and $$x^2 + x^2y^4 = 328. \quad (2)$$

From (1), $$2x^2y^2 = 72. \quad (3)$$

Sum, $$x^2 + 2x^2y^2 + x^2y^4 = 400.$$

Square root, $$x + xy^2 = 20. \quad (4)$$

Subtract (3) from (2), $$x^2 - 2x^2y^2 + x^2y^4 = 256.$$

Square root, $x - xy^2 = 16$; or, $xy^2 - x = 16$. (5)

(5) from (4), $$2x = 4, \text{ and } x = 2.$$

Now, (1) becomes $2y = 6$, and $y = 3$.

Hence 2, 6, 18, are the numbers.

9. Here, $$x + xy + xy^2 = 13, \quad (1)$$

and $$(x + xy^2)\, xy = 30. \quad (2)$$

From (1), $$x + xy^2 = 13 - xy. \quad (3)$$

From (2), $$x + xy^2 = \frac{30}{xy} \quad (4)$$

Equating (3) and (4), $$13 - xy = \frac{30}{xy}$$

$$x^2y^2 - 13xy = -30.$$

Whence, $xy = 10$, or 3.

It cannot be 10, and correspond with (2), therefore take 3.

Then (2) becomes $$x + xy^2 = 10. \quad (5)$$

From (5), $$x = \frac{10}{1 + y^2}.$$

But $xy = 10$, and $$x = \frac{3}{y};$$

Equating the values of $x$, and $3y^2 + 3 = 10y$;

whence, $$y = 3, \text{ and } x = 1.$$

10. Here, $x + xy^2 = 52$; $x^2y^2 = 100$; and $xy = 10$.

Then, $x + xy^2 = 52$; $x = \dfrac{52}{1+y^2}$; and $x = \dfrac{10}{y}$;

whence, $\dfrac{10}{y} = \dfrac{52}{1+y^2}$; $10 + 10y^2 = 52y$

$5y^2 - 26y = -5$;

whence, $y = 5$.

But, $x = \dfrac{10}{y} = \dfrac{10}{5} = 2$.

11. Let $x$, $\sqrt{xy}$, $y$, represent the numbers.

Then, $x + \sqrt{xy} + y = 31 = a$; (1)

and $x^2 + y^2 = 626 = b$. (2)

From (1), $x + y = a - \sqrt{xy}$; (3)

square (3) $x^2 + 2xy + y^2 = a^2 - 2a\sqrt{xy} + xy$;

$x^2 + y^2 = a^2 - 2a\sqrt{xy} - xy$. (4)

By equating (2) and (4), and transposing, we obtain

$xy + 2a\sqrt{xy} = a^2 - b$;

$xy + 2a\sqrt{xy} + a^2 = 2a^2 - b$.

Sq. root, $\sqrt{xy} + a = \pm\sqrt{2a^2 - b}$.

$a = 31$, $a^2 = 961$, $2a^2 - b = 1922 - 626 = 1296$

Whence, $\sqrt{xy} + 31 = \pm 36$; $\sqrt{xy} = 5$, $xy = 25$. (5)

This placed in (3), and $x + y = 26$. (6)

From (5) and (6) we obtain $x = 1$, and $y = 25$.

12. Here, $\left.\begin{array}{ll} x + \sqrt{xy} + y = 14, & (1) \\ x^2 + xy + y^2 = 84. & (2) \end{array}\right\}$ Solution the same as Ex. 1, in the book.

13. Let $x$, $xy$, $xy^2$, $xy^3$, represent the numbers.

$$xy^3 - xy = 24; \qquad (1)$$

$$x + xy^3 : xy + xy^2 :: 7 : 3; \qquad (2)$$

or, $$1 + y^3 : y(1 + y) :: 7 : 3.$$

Dividing the first and second terms by $(1 + y)$,

$$1 - y + y^2 : y :: 7 : 3;$$

$$3 - 3y + 3y^2 = 7y;$$

or, $$3y^2 - 10y + 3 = 0;$$

$$36y^2 - (\ ) + 100 = 100 - 36 = 64;$$

$$6y - 10 = \pm 8, \text{ and } y = 3, \text{ or } \tfrac{1}{3}.$$

Substituting the value of $y$ in (1), and $x = 1$.

14. Let $x$, $xy$, $xy^2$, $xy^3$, represent the numbers.

$$x + xy + xy^2 + xy^3 = y + 1;$$

and, $$x = \tfrac{1}{10}.$$

Then, $$\tfrac{1}{10} + \tfrac{1}{10}y + \tfrac{1}{10}y^2 + \tfrac{1}{10}y^3 = y + 1;$$

$$1 + y + y^2 + y^3 = 10y + 10;$$

$$y^2 + y^3 = 9(y + 1);$$

That is, $$(1 + y)y^2 = 9(y + 1);$$

$$y^2 = 9, \text{ and } y = 3.$$

Hence, $\frac{1}{10}$, $\frac{3}{10}$, $\frac{9}{10}$, $\frac{27}{10}$ are the numbers.

15. The formula for compound interest is

$$(1 + r)^n p = \text{the amount}.$$

An equation in which $r =$ the rate per cent, $n =$ the number of years, and $p =$ the principal.

In this example, $r = .07$, $p = 500$, and $n = 4$.

$$\text{Whence, } (1.07)^4 = (1.1449)(1.1449) = p = \begin{array}{r} 1.31079601 \\ 500 \\ \hline \$655.39800500 \end{array}$$

## PROPORTION.

(**276**, page 288.)

5. Let $x$ and $y$ represent the numbers.

Then, $\quad x - y : x + y :: 2 : 9;$ (1)

$x + y : xy :: 18 : 77.$ (2)

From (1), $\quad 2x : 2y :: 11 : 7;$ (Prop. VII.)

$x : y :: 11 : 7;$

and $\quad x = \dfrac{11y}{7}.$

Place this value of $x$ in (2), and we have

$$\frac{11y}{7} + y : \frac{11y^2}{7} :: 18 : 77,$$

$$18y : 11y^2 :: 18 : 77,$$

by multiplying the first couplet by 7. Now divide by $y$, and

$$18 : 11y :: 18 : 77.$$

Inverting the means,

and, $\quad 18 : 18 :: 11y :: 77.$

Divide the first couplet by 18, and the last by 11;

$$1 : 1 :: y : 7; \text{ and } y = 7.$$

But $\quad x = \dfrac{11y}{7} = 11.$

6. Let $x$ and $y$ represent the numbers.

Then, $x + 4 : y + 4 :: 3 : 4$; (1)

$x - 4 : y - 4 :: 1 : 4$. (2)

From (1), $4x + 16 = 3y + 12$;

$$x = \frac{3y - 4}{4}. \quad (3)$$

This value of $x$ placed in (2), gives

$$\frac{3y - 4}{4} - 4 : y - 4 :: 1 : 4.$$

then, $3y - 4 - 16 = y - 4$;

$$2y - 16 = 0; \text{ or } y = 8, \text{ and } x = \frac{24 - 4}{4} = 5, \textit{ Ans.}$$

7. Let $x$ and $y$ represent the parts.

Then, $x + y = 16$; (1)

$xy : x^2 + y^2 :: 15 : 34$; (2)

whence, $2xy : x^2 + y^2 :: 30 : 34$.

Prop. VIII., $x^2 - 2xy + y^2 : x^2 + 2xy + y^2 :: 4 : 64$.

Square root, $x - y : x + y : 2 : 8$;

or, (1) $x - y : 16 : 2 : 8$;

whence, $x - y = 4$;

but, (1) $x + y = 16$.

Whence, 10 and 6 are the numbers.

8. Let $x + y =$ the greater number,

and $x - y =$ the less.

Then, $x^2 - y^2 = 320$; (1)

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3;$$

$$(x - y)^3 = \underline{x^3 - 3x^2y + 3xy^2 - y^3}.$$

Difference of their cubes, $6x^2y + 2y^3$.

$2y = 8y^3 =$ the cube of their difference.

Hence, $6x^2y + 2y^3 : 8y^3 :: 61 : 1;$ (2)

or, $3x^2 + y^2 : 4y^2 :: 61 : 1.$

$3x^2 + y^2 = 244y^2;$

$x^2 = 81y^2.$

This value of $x^2$ in (1).

$81y^2 - y^2 = 320;$

$80y^2 = 320,\ y^2 = 4,$ an d $y = 2.$

But, $x = 9y;$ whence, $x = 18.$

$x + y = 20$, and $x - y = 16$, *Ans.*

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## MISCELLANEOUS EXAMPLES.

(**216**, page 298.)

13. As $x$ and $y$ are raised to the same power, it is obvious that the fractions must involve the like power of $x$ and $y$. We will therefore assume $(x + y)$ to be one factor; and then, by trial,—that is, dividing $x^5 + y^5$ by it,—we find $x^4 - x^3y + x^2y^2 - xy^3 + y^4$ to be the other factor.

14. Here, the minus sign is between the powers $x^5 - y^5$. Therefore, half the exponents must be the powers in the required factors; $(x^{\frac{5}{2}} + y^{\frac{5}{2}})$ into $(x^{\frac{5}{2}} - y^{\frac{5}{2}})$; that is, the product of sum and difference.

15.

$$
\begin{array}{l}
a^{\frac{5}{3}} - a^{\frac{4}{3}}b + ab^2 - a^{\frac{2}{3}}b^3 + a^{\frac{1}{3}}b^4 - b^5 \\
\underline{a^{\frac{1}{3}} + b} \\
a^2 - a^{\frac{5}{3}}b + a^{\frac{4}{3}}b^2 - ab^3 + a^{\frac{2}{3}}b^4 - a^{\frac{1}{3}}b^5 \\
\underline{\quad\ + a^{\frac{5}{3}}b - a^{\frac{4}{3}}b^2 + ab^3 - a^{\frac{2}{3}}b^4 + a^{\frac{1}{3}}b^5 - b^6} \\
a^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - b^6
\end{array}
$$

or, $a^2 - b^6$, *Ans.*

16. Find the factors of $a^8 - b^4$.

We will take half of each exponent, and form two factors—one with the plus sign, the other with the minus; thus:

$$(a^4 + b^2),\ (a^4 - b^2).$$

Now this last factor we can separate into two others; thus, $(a^4 - b^2)$ is equal to $(a^2 + b)\ (a^2 - b)$.

The factors $(a^4 + b^2)$, $(a^2 + b)$, and $(a^2 - b)$, may also be other factors.

17. Find the greatest common factor of $(a^4 - 1)\ (a^5 + a^3)$ and $(a^6 + 1)$.

$$a^4 - 1 = (a^2 + 1)\ (a^2 - 1).$$
$$a^5 + a^3 = (a^2 + 1)a^3.$$
$$a^6 - 1 = (a^2 + 1)\ (a^4 - a^2 + 1).$$

Thus we see that $(a^2 + 1)$ is a factor common to all the terms, and it is therefore the greatest factor.

19. $$S = \frac{a^2}{(a+b)^2} + \frac{b}{a+b} + \frac{ab}{(a+b)^2}.$$

Multiply each member by $(a + b)^2$.

Then, $(a + b)^2S = a^2 + ab + b^2 + ab = a^2 + 2ab + b^2 = (a + b)^2$.

Divide by $(a + b)^2$, and $S = 1$, *Ans.*

20. $$S = \frac{x}{x^2 - y^2} + \frac{y}{x+y} + \frac{1}{x-y}.$$

Multiply each term by $(x^2 - y^2)$,

and, $(x^2 - y^2)S = x + (x - y)y + x + y = 2x + xy - y^2 + y$.

Whence, $$S = \frac{2x + xy - y^2 + y}{x^2 - y^2},\ \textit{Ans.}$$

21. Remember that, in dividing exponential quantities, as $a^4$ by $a^2$, we subtract the exponents.

$$17a^2)51a^2(3, \text{ one factor in the quotient.}$$

$$\begin{array}{lr} & m+n \\ & \underline{m-n} \\ \text{Difference,} & 2n \end{array}$$

Whence, $3(c-1)^{2n}$ is the required quotient.

22. Dividing by $1-a$, we have $-5+6(1-a)=1-6a$, *Ans.*

23. Here, $m-n$ is equal to $(\sqrt{m}+\sqrt{x})(\sqrt{m}-\sqrt{x})$, and we are required to divide by this last factor. Then the other factor, $\sqrt{m}+\sqrt{n}$, must be the quotient.

24.
$$\begin{array}{r|l} a^2 + \quad ab + \ b^2 & (a+\sqrt{ab}+b \\ \underline{a^2 + a\sqrt{ab} + ab} & \overline{a-\sqrt{ab}+b}, \text{ quotient.} \\ -a\sqrt{ab}+b^2 & \\ \underline{-a\sqrt{ab}-ab-b\sqrt{ab}} & \\ ab+b\sqrt{ab}+b^2 & \\ ab+b\sqrt{ab}+b^2 & \end{array}$$

25. We assume that $(x-y)$ is one factor.

$$\begin{array}{r|l} x^2y-xy^2-xy+x-y & (x-y \\ \underline{x^2y-xy^2 \qquad\qquad\quad} & \overline{xy+1} \\ -xy+x-y & \\ \underline{x-y} & \\ -xy, & \text{remainder.} \end{array}$$

Whence, $(xy+1)-\dfrac{xy}{x-y}$ is the other factor.

26. Here, as $4az$ is common to all the terms in the first quantity, and not common in the other quantity, we shall divide the first quantity by it; and that quantity thus reduces to

$$a^2 - 3mz + 2bc.$$

In the second quantity, $5mx$ is a factor common to all the terms, and therefore we will cast it out by division; and the second quantity will thus reduce to

$$a^2 - 3mz + 2bc;$$

the same as the other. Hence, this last quantity is a divisor common to both quantities.

27.
$$\begin{array}{r|lll} 2ax & 3a^2x & 4axy & 16ax^3 \\ \hline 2 & 3a & 2y & 8x^2 \\ \hline & 3a & y & 4x^2 \end{array}$$

$$4ax \times 3a \times y \times 4x^2 = 48a^2x^3y, \; Ans.$$

28. If we multiply the third quantity by $a$, the product, $a^3c - 9acm^4$, contains all the factors of the other two quantities; hence this product is the least common multiple.

29. Observe that the numerator is equal to $(a - b)\sqrt{ab}$, and that the denominator is $(a - b)^2$. Therefore the greatest common factor is $(a - b)$. Dividing the terms of the fraction by this factor, we obtain

$$\frac{\sqrt{ab}}{a-b}, \; Ans.$$

30. Observe that $\dfrac{1}{(\sqrt{a}-\sqrt{b})^{-1}} = (\sqrt{a}-\sqrt{b})^1$.

Whence, $m(\sqrt{a}-\sqrt{b})^2$, or $m(a-2\sqrt{ab}+b)$, is the result sought.

31. Here the difference between two quantities is demanded.

Represent that difference by D, and we have the equation,

$$D = \frac{c}{c-1} - \frac{c^3}{c^3-1} - \frac{2c}{c^4-1}.$$

Multiply each member by $c^2-1$, and

$$(c^2-1)D = c^2+c-c^3-\frac{2c}{c^2+1} = c - \frac{2c}{c^2+1};$$

$$(c^2-1)(c^2+1)D = c^3-c = (c^2+1)c;$$

$$(c^2+1)D = c.$$

Whence, $D = \dfrac{c}{c^2+1}$.

32. $$D = \frac{1}{x^3}+\frac{1}{x^2}+\frac{x-1}{x^2+1}-\frac{1}{x}-\frac{1}{x^2+1}.$$

$$Dx^3 = 1+x+\frac{(x-1)x^3}{x^2+1}-x^2-\frac{x^3}{x^2+1}.$$

$$Dx^3(x^2+1) = 1+x+x^2+x^3+x^4-x^3-x^4-x^2-x^3.$$

$$D = \frac{1+x-x^3}{x^3(x^2+1)},\ \textit{Ans.}$$

34. $\dfrac{a^2x^2}{a^2-x^2} \div \dfrac{ax+x^2-ax}{a+x}$, or by $\dfrac{x^2}{a+x} = \dfrac{a^2}{a-x}$, *Ans.*

(299, 300)

35. Let $a + x = P$, and $a - x = Q$.

Then, $$\frac{\frac{P}{Q} + \frac{Q}{P}}{\frac{P}{Q} - \frac{Q}{P}}$$

Multiply the terms of the fractions by P Q.

$$\frac{P^2 + Q^2}{P^2 - Q^2}$$

$$P^2 = a^2 + 2ax + x^2$$
$$Q^2 = a^2 - 2ax + x^2$$
$$P^2 - Q^2 = 4ax$$
$$P^2 + Q^2 = 2(a^2 + x^2);$$

whence $\frac{a^2 + x^2}{2ax}$ is the result sought.

36. Multiply each quantity by $\sqrt{ab}$; then we must divide.

$$2\sqrt{ab} - a - b \div \sqrt{a} - \sqrt{b}.$$

That is, divide $- \frac{a - 2\sqrt{ab} + b}{\sqrt{a} - \sqrt{b}} = \sqrt{a} + \sqrt{b} =$ *Ans.*

37. $(a + b)^2 = a^2 + 2ab + b^2$ From $a^2 - b^2$
$(a - b)^2 = a^2 - 2ab + b^2$ Sub. $a^2 - 2ab + b^2$.

Difference, $4ab$. Diff. $2ab - 2b^2$.

Fractions, $\frac{4ab}{2ab - 2b^2} = \frac{2a}{a - b}$, *Ans.*

38. $$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Difference, $6a^2b + 2b^3$

$$(1 + b)^3 = 1 + 3b + 3b^2 + b^3$$
$$(1 - b)^3 = 1 - 3b + 3b^2 - b^3$$

Difference, $6b + 2b^3$.

Fraction, $\frac{6a^2b + 2b^3}{6b + 2b^3} = \frac{3a^2 + b^2}{3 + b^2}$.

40. Let $b = \sqrt{-1}$; then we must cube $ab$, which gives $a^3b^3$. But $b = \sqrt{-1}$, $b^2 = -1$, $b^3 = -1\,a^3(-\sqrt{-1}) = -a^3\sqrt{-1}$, *Ans.*

41. This is easily raised to the 9th power by French's method of coefficients.

$$C_1 = 2^9 = 512$$

$$C_2 = \frac{512 \times 9 \times 3}{2} = 256 \times 27$$

$$C_3 = \frac{256 \times 27 \times 8 \times 3}{2 \times 2} = 512 \times 81$$

$$C_4 = \frac{512 \times 81 \times 7 \times 3}{3 \times 2} = 256 \times 567$$

$$C_5 = \frac{256 \times 567 \times 6 \times 3}{4 \times 2} = 576 \times 567$$

$$C_6 = \frac{576 \times 567 \times 5 \times 3}{5 \times 2} = 567 \times 864$$

$$C_7 = \frac{567 \times 864 \times 4 \times 3}{6 \times 2} = 567 \times 864$$

$$C_8 = \frac{567 \times 864 \times 3 \times 3}{7 \times 2} = 729 \times 432$$

$$C_9 = \frac{729 \times 432 \times 2 \times 3}{8 \times 2} = 729 \times 162$$

$$C_{10} = \frac{729 \times 162 \times 1 \times 3}{9 \times 2} = 729 \times 27$$

whence, $512a^9 + 256 \times 27a^8b + C_3a^7b^2 + C_4a^6b^3$, &c., to $C_{10}$.

42. From $(x + \sqrt{a})^2 = x^2 + 2x\sqrt{a} + a$

Subtract $(x - \sqrt{a})^2 = x^2 - 2x\sqrt{a} + a$

$$4x\sqrt{a} = 2a\sqrt{ab}$$
$$2x = a\sqrt{b}$$

43. Divide each term by $\sqrt{3a}$.

Then, $$\frac{x\sqrt{m}}{m} = 1 + \frac{\sqrt{4c^2}}{m}$$
$$x\sqrt{m} = m + 2c;\ x = \frac{m + 2c}{\sqrt{m}}.$$

44. Multiply by $x$, and

$$b\sqrt{a} = \frac{(b + \sqrt{a})x}{b\sqrt{a}};\ \text{whence, } x = b\sqrt{a}.$$

45. By transposing the 2d term, we have

$$\frac{4}{x - 1} = \frac{x + 1}{2};\ x^2 - 1 = 8;$$

whence, $x^2 = 9$, and $x = \pm 3$.

46. Divide by 5; then,

$$\frac{x^2 + 1}{5x + 5} = x - 1;\ \text{or, } \frac{x^2 + 1}{5} = x^2 - 1.$$
$$x^2 + 1 = 5x^2 - 5$$
$$-4x^2 = -6;\ 2x = \sqrt{6};\ x = \pm\tfrac{1}{2}\sqrt{6}.$$

47. Clearing of fractions, and $x^2 = x^4 - 1$;

or, $$x^4 - x^2 = 1$$
$$4x^4 - (\ ) + 1 = 5$$
$$2x^2 - 1 = \sqrt{5};\ 4x^2 = 2 \pm 2\sqrt{5}$$
$$2x = \pm\sqrt{2 \pm 2\sqrt{5}}$$
$$x = \pm\tfrac{1}{2}\sqrt{2 \pm 2\sqrt{5}}.$$

8. Given $\sqrt{5x-9} = \dfrac{4x}{\sqrt{5x-9}}$;

whence, $5x - 9 = 4x$; and $x = 9$.

49. Expand

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$
$$(1-x)^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$$

Sum, $2 + 20x^2 + 10x^4 = 242$

$$x^4 + 2x^2 = 24$$
$$x^4 + 2x^2 + 1 = 25: \ x^2 + 1 = \pm 5$$
$$x^2 = 4, \text{ or } -6$$
$$x = \pm 2 \pm \sqrt{-6}.$$

50. Here, $C_1 = (\sqrt{2})^6 = 2^3 = 8$

$$C_2 = \frac{8 \times 6 \times \sqrt{3}}{\sqrt{2}} = 24\sqrt{6}$$

$$C_3 = \frac{24\sqrt{6} \times 5 \times \sqrt{3}}{2\sqrt{2}} = 180$$

$$C_4 = \frac{180 \times 4 \times \sqrt{3}}{3\sqrt{2}} = 120\sqrt{6}$$

$$C_5 = \frac{120\sqrt{6} \times 3 \times \sqrt{3}}{4\sqrt{2}} = 270$$

$$C_6 = \frac{270 \times 2 \times \sqrt{3}}{5\sqrt{2}} = 54\sqrt{6}$$

$$C_7 = \frac{54\sqrt{6}\sqrt{3}}{6\sqrt{2}} = 27;$$

whence, $8 + 24\sqrt{6} + 180 +$, &c.

51. Divide numerator by denominator, and we have

$$y - 3 = 7 - y;$$
$$2y = 10; \; y = 5.$$

52. Divide numerator by denominator, and

$$5x - 3 = 7 - 5x; \; 10x = 10 -; \; x = 1.$$

53. Double every term, and

$$x + 4 = x\left(\frac{6}{x} - 1\right) + \frac{15 - x}{3}$$
$$x + 4 = 6 - x + 5 - \tfrac{1}{3}x$$
$$2x = 7 - \tfrac{1}{3}x; \; 6x = 21 - x;$$
$$7x = 21; \; x = 3.$$

54. Multiply the factors in the second member.

$$x^3 + 2x^2 + x = x^3 + 3x^2 - x^2 - 3x + 32;$$
$$4x = 32; \; x = 8.$$

55. Expanding the binomials;

$$(x + 1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$
$$(x - 1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$$

Difference, $12x^5 + 40x^3 + 12x = 1344x.$

Dividing by $2x$, and we have

$$6x^4 + 20x^2 + 6 = 672;$$
$$3x^4 + 10x^2 = 333; \; \text{assume } x^2 = \frac{y}{3}.$$

Then our equation becomes

$$\frac{y^2}{3} + \frac{10y}{3} = 333; \; y^2 + 10y = 999;$$
$$y^2 + (\;) + 25 = 1024;$$
$$y + 5 = \pm 32; \text{ and } y = 27, \text{ or } -37.$$

Then, $x^2 = 9$, or $x^2 = -\frac{37}{3}$; $x = \pm 3$, or $x = \pm\sqrt{-\frac{37}{3}}$, *Ans.*

56. Add the two equations, and we shall have

$$6x + 6y = 36$$

$$\left.\begin{aligned} x + y &= 6 \\ 5x + y &= 26 \end{aligned}\right\} \text{Difference, } 4x = 20;\ x = 5.$$

57. Add the two given equations, and

$$(a + c)x + (a + c)y = ac^3 + 2a^2c^2 + a^3c$$
$$= ac\,(c^2 + 2ac + a^2).$$

Divide by $(a + c)$, and

$$x + y = a\,c(a + c)$$

Multiply by $a$, and $ax + ay = a^2c(a + c)$.

But, $ax + cy = 2a^2c^2$.

By subtraction, $(a - c)y = a^3c - a^2c^2 = a^2c(a - c)$.

By division, $y = a^2c$.

58. Add $2z$ to each side of the first equation, $3y$ to the 2d, and $4x$ to the 3d.

Then,
$$x + y + z = 4 + 2z$$
$$2x + 2y + 2z = 6 + 3y$$
$$3x + 3y + 3z = 6 + 4x.$$

Assume $x + y + z = S$; then

$$S = 4 + 2z \qquad (1)$$
$$2S = 6 + 3y \qquad (2)$$
$$3S = 6 + 4x. \qquad (3)$$

Multiply (1) by 6; (2) by 4; and (3) by 3; then we shall have

$$6S = 24 + 12z$$
$$8S = 24 + 12y$$
$$9S = 18 + 12x.$$

By addition, $23S = 66 + 12S$;

or, $11S = 66$; whence, $S = 6$.

This value put in (1), gives $6 = 4 + 2z$; $z = 1$.

Equation (2) will give $y$, and (3) will give $x$.

59. Adding the equations, and we have

$$3n + 3x + 3y + 3z = 12a - 30m;$$

or, $\quad n + x + y + z = 4a - 10m. \qquad (1)$

But, $\quad \underline{n + x + y \qquad = 3a - 6m.} \qquad (2)$

Subtracting (2) from (1), and $\quad z = a - 4m.$

From (1) subtract each of the given equations, and we shall have the values of $z$, $y$, $x$, and $n$.

60. Double the first equation and add to the second, and

$4x^2 + 2x = 42$; or $2x^2 + x = 21$; whence, $x = 3$.

61. The equations are homogeneous; therefore put $x = vy$.

Then the equations become, $v^2y^2 + 3y^2 = 28; \qquad (1)$

and, $\qquad v^2y^2 + 2vy^2 = 35. \qquad (2)$

$$y^2 = \frac{28}{v^2+3},\ y^2 = \frac{35}{v^2+2v}; \qquad (3)$$

$$28v^2 + 56v = 35v^2 + 105;$$

$$-7v^2 + 56v = 105.$$

Divide by $-7$, and $v^2 - 8v = -15$; $v^2 - (\ ) + 16 = 1$;

$$v - 4 = \pm 1,\ v = 5, \text{ or } 3.$$

But, $y^2 = \dfrac{35}{v^2+2v} = \dfrac{35}{25+10} = 1$; or, $\dfrac{35}{9+6} = \frac{7}{3}$.

That is, $y = \pm 1$, or $\pm\sqrt{\frac{7}{3}} = \pm\frac{1}{3}\sqrt{21}$.

Again, $x = vy = 5$, or $3$.

62. $x + \sqrt{xy}$ is equal to $(\sqrt{x} + \sqrt{y})\sqrt{x}$;

$y + \sqrt{xy}$ is equal to $(\sqrt{x} + \sqrt{y})\sqrt{y}$.

Now, if we divide the first quantity by the second, the value of the quotient will be

$$\frac{\sqrt{x}}{\sqrt{y}} = \frac{15}{10} = \frac{3}{2}.$$

By squaring, $x = \frac{9}{4}y.$

By subtracting the given equation, we have

$$x - y = 5.$$

Then, $\frac{9}{4}y - y = 5;$

$$9y - 4y = 5 \times 4;$$

$$5y = 5 \times 4; \text{ and } y = 4.$$

63. Double the second equation, and add it to the first. Then we shall have,

$$x^4 + 2x^2y + y^2 = 196.$$

Square root, $x^2 + y = \pm 14.$ (1)

Sublimating the product, and

$$x^4 - 2x^2y + y^2 = 16.$$

Square root, $x^2 - y = 4.$ (2)

Adding (1) and (2), $2x^2 = 18$, and $x = \pm 3.$

Subtracting (2) from (1), $2y = 10$, and $y = 5.$

64. Let $15x =$ the distance between A and B.

A traveled $\frac{1}{3}$ of $15x$, or $5x$, and B traveled $6x$.

Then they were $15x - 11x = 4x = 16$ miles apart.

Reducing, $x = 4$, and $15x = 60$, *Ans.*

65. Let $x =$ the greater number, and $y$ the less.

By the conditions, $x + y = 80,$

and $y - x + y : 2x - y :: 1 : 7$

$$14y - 7x = 2x - y$$

$$15y = 9x, \text{ and } 5y = 3x$$

$$3x + 3y = 80 \times 3$$

$$8y = 80 \times 3, \text{ and } y = 30.$$

66. Let $x =$ the number.

$$\frac{x}{4} - \frac{x}{5} = \frac{1}{40};\ 5x - 4x = \frac{1}{2},\ \text{and } x = \frac{1}{2}.$$

67. Let $100x + 10x + x =$ the number.

$$\begin{array}{l} 4x + \ \ 4x + 4x \\ \hline 96x + \ \ 6x - 3x = 99x = 297, \text{ and } x = 3. \end{array}$$

Whence, $333 =$ the number.

68. Let $x =$ the number.

Then, $x + 1 : x + 4 :: x^2 : x^3$

or, $x + 1 : x + 1 :: 1 : x$

whence, $x^2 + x = x + 4$, $x^2 = 4$, and $x = 2$, *Ans.*

69. Let $x$ and $y$ represent the capacities of the casks.

Then, $2x + 10y = 50$ (1)

And, $6x + 5y = 50$ (2)

The double of (2), $12x + 10y = 100$ (3)

Subtract (1) from (3), $10x = 50$, and $x = 5$.

70. Let $x =$ the income of each.

One spends $\frac{9}{10}x$.

The other spends $\frac{9x}{10} + 150 = x + 20$.

whence, $\frac{x}{10} = 130$, and $x = 1300$.

71. Let $x =$ the number in each.

After selling, he has in one flock $x - a$ sheep,

and in the other, $x - b$ sheep.

Then, $3x - 3a = x - b$

$2x = 3a - b$, and $x = \frac{1}{2}(3a - b)$.

(302, 303)

72. Let $x^3$ and $y^3$ represent the numbers.

Then, $x^3 + y^3 = 72$ (1)

and, $x + y = 6$ (2)

Dividing (1) by (2), $x^2 - xy + y^2 = 12$ (3)

Squaring (2), $x^2 + 2xy + y^2 = 36$ (4)

(3) from (4), and $3xy = 24$, and $xy = 8$ (5)

(5) from (3), $x^2 - 2xy + y^2 = 4$ (6)

$x - y = 2$ (7)

But (2), $x + y = 6$ (8)

Adding (7) and (8), $2x = 8$, $x = 4$, and $x^3 = 64$, one number.

Sub. (7) from (8), $2y = 4$, $y = 2$, and $y^3 = 8$, the other number.

73. Let $x$ and $y$ represent the numbers.

Then, $(x^2 + y^2)(x + y)\frac{1}{2} = 2336$

$(x^2 - y^2)(x - y)\frac{1}{2} = 576$

$x^3 + x^2y + xy^2 + y^3 = 2336$ (1)

$x^3 - x^2y - xy^2 + y^3 = 576.$ (2)

By addition, $2x^3 + 2y^3 = 2912$

$x^3 + y^3 = 1456.$ (3)

By subtraction, $2x^2y + 2xy^2 = 1760$

$x^2y + xy^2 = 880.$ (4)

Three times (4) added to (3), produces

$x^3 + 3x^2y + 3xy^2 + y^3 = 4096$, cubes.

Cube root, $x + y = 16.$ (5)

Factoring (4), and $xy(x + y) = 880.$ (6)

Dividing (6) by (5), $cy = 55.$ (7)

(3) divided by (5), gives $x^2 - xy + y^2 = 91.$ (8)

Subtract (7) from (8) $x^2 - 2xy + y^2 = 36.$ (9)

Square root, $x - y = 6.$ (10)

Add (5) and (10), $2x = 22$; and $x = 11$,

74. Let $x$ and $y$ represent the numbers.

$$xy(x+y) = 84 \quad = a \qquad (1)$$

$$x^2y^2(x^2+y^2) = 3600 = b \qquad (2)$$

From (1), $$x+y = \frac{a}{xy}$$

Squaring, $$x^2+y^2 = \frac{a^2}{x^2y^2} - 2xy. \qquad (3)$$

From (2), $$x^2+y^2 = \frac{b}{x^2y^2}. \qquad (4)$$

Equating the 2d members of (3) and (4),

and, $$\frac{a^2}{x^2y^2} - 2xy = \frac{b}{x^2y^2}$$

$$a^2 - 2x^3y^3 = b;$$

whence, $$xy = \left(\frac{a^2-b}{2}\right)^{\frac{1}{3}} = 12.$$

This value of $xy$ substitued in (1), and reduced, gives

$$x+y = 7$$

And substituted in (4), gives $x^2+y^2 = 25$

$$2xy = 24$$

Subtracting, $$x^2 - 2xy + y^2 = 1$$

$$x - y = 1$$

$$x + y = 7$$

$$2x = 8;$$

whence, $$x = 4;\ y = 3.$$

75. Let $x$ represent the price of a duck,
and $y$ the price of a turkey.

Then, $$15x + 12y = 105, \qquad (1)$$

and, $$\frac{18}{x} - 2 = \frac{20}{y}. \qquad (2)$$

Divide (2) by 2, &c., $9y - xy = 10x;\ y = \dfrac{10x}{9-x}.$

Whence (1) becomes $15x + \frac{120x}{9-x} = 105$;

$$x + \frac{8x}{9-x} = 7$$

$$9x - x^2 + 8x = 63 - 7x$$

$$x^2 - 24x = -63$$

$$x^2 - (\ ) + 144 = 81$$

$$x - 12 = \pm 9;$$

whence, $x = 21$; or, 3.

But $x$ cannot equal 21 in this problem, because $\frac{18}{x}$ represents a number of turkeys for 18 shillings; but if $x = 21$, $\frac{18}{21}$, *is a fraction.*

76. $x + y + z = 12$ (1)

$$\frac{x+y}{3} = \frac{y+z}{5} \quad (2)$$

$$y - x = z - y \quad (3)$$

From (3), we obtain $2y = x + z$.

Substitute this value of $(x + z)$ in (1), and we have

$$3y = 12;\ y = 4.$$

Then, $x + z = 8$.

From (2), $5x + 8 = 3z$;

$$3x + 3z = 24.$$

Sum, $8x = 16$: and $x = 2$.

77. Let $x$ represent a side of the square in rods; then $\frac{x^2}{160}$ = the number of acres; $4x$ = the length of fence inclosing the square; and $40x$ = the number of acres.

Whence, $\frac{x^2}{160} = 40x$; $x = 6400$ rods $= 20$ miles.

78. Let $x$ represent the number of men on a side.

Then $x^2 + 92$, the number in the regiment.

Also, $(x + 2)^2 - 100 =$ the number in the regiment.

Whence,

$$x^2 + 4x + 4 - 100 = x^2 + 92$$
$$4x - 96 = 92$$
$$x - 24 = 23;\ x = 47;\ \overline{47}^2 + 92 = 2301,\ \textit{Ans.}$$

79. Let $2x^2 =$ the number of bees.

After the 1st flight $2x^2 - x$ bees were left.

$$\frac{8}{9} \text{ of } 2x^2 = \frac{16x^2}{9}.$$

Whence, $2x^2 - x - \frac{16x^2}{9} = 2$

$$18x^2 - 9x - 16x^2 = 18$$
$$2x^2 - 9x = 18$$
$$16x^2 - (\ \ ) + 81 = 144 + 81 = 225$$
$$4x - 9 = \pm 15;\ \text{and } x = 6, \text{ or } -\tfrac{3}{2}.$$

Whence, $2x^2 = 72$, *Ans.*

80. Let $x$ represent the distance above the ground; then $56 - x =$ the part broken off, which forms the hypothenuse of a right angled triangle; 12 being the base, and $x$ the perpendicular.

Whence, $(56 - x)^2 = 144 + x^2$

$$\overline{56}^2 - 112x = 144$$
$$112x = \overline{56}^2 - \overline{12}^2 = 68 \times 44.$$

Dividing by 4, and $28x = 68 \times 11$

$$7x = 17 \times 11 = 187;\ x = 26\tfrac{5}{7}.$$

81. Let $x$ = the greater part, and $y$ = the less.

$$x + y = 20 \qquad (1)$$

$$x^2 - 2y = 2y^2. \qquad (2)$$

From (1), $x^2 = 400 - 40y + y^2$.

" (2), $x^2 = 2y + 2y^2$.

Whence, $y^2 + 42y = 400$

$$y^2 + (\ ) + \overline{21}^2 = \overline{21}^2 + 400 = 841$$

$$y + 21 = \pm 29, \text{ and } y = 8; \text{ whence, } x = 12.$$

82. Let $(x - y)$, $x$, and $(x + y)$, represent the numbers.

$$3x = 27; \ x = 9;$$

$$x^2 - y^2 = 77.$$

That is, $81 - 77 = y^2$; whence, $y = 2$.

83. Let $x$ = the width of the walk.

Then, $10 - x$ = the length of the walk on the side;

$8 - x$ = the length on the end.

The contents, on the length, is $10x - x^2$;

on the end, $8x - x^2$;

in the corner, $x^2$.

Contents of the walk, $18x - x^2 = \frac{80}{6} = \frac{40}{3}$;

or, $x^2 - 18x + 81 = 81 - \frac{40}{3} = 67\frac{2}{3}$

$$x - 9 = 8.\ 226 +; \ x = 17.\ 226 + \text{ feet.}$$

In the 1st edition the answer corresponds to a walk outside of the garden.

84. Let $ax$ and $bx$, be the numbers.

Then, $ax + c : bx + c :: 5 : 6$

$6ax + 6c = 5bx + 5c$; $(5b - 6a)x = c$

$x = \frac{c}{5b - 6a}$; $ax = \frac{ac}{5b - 6a}$, *Ans.*

85. Let $x$ represent the cost of the horse;

and $144 - x =$ the gain;

But 100, gives $x$.

Therefore, $x : 144 - x :: 100 : x$

$x^2 = 14400 - 100x$

$x^2 + 100x + \overline{50}^2 = 14400 + \overline{50}^2 = 16900$

$x + 50 = \pm 130$; and $x = 80$, *Ans.*

86. Let $5x$ and $8x$, be the numbers.

Then, $5x + 200 : 8x + 120 :: 5 : 4$.

Divide the 1st and 3d terms by 5; and the 2d and 4th by 4.

Whence, $x + 40 : 2x + 30 :: 1 : 1$

$2x + 30 = x + 40$; $x = 10$; whence, $5x = 50$, and $8x = 80$.

87. This is illustrated in the text book; we take it where the book leaves it. Make $x^2y^2 = \text{P}$, and clear of fractions;

$4\text{P}^2 = b^2 - a^2\text{P}^2$

$4\text{P}^2 + a^2\text{P} = b^2$ $\qquad a^2 = 81$

$4 \times 16\text{P}^2 + (\ \ ) + a^4 = a^4 + 16b^2$ $\qquad a^4 = 6561$

$8\text{P} + a^2 = \pm\sqrt{a^4 + 16b^2}$ $\qquad b^2 = \overline{820}^2 = 672400$

$8\text{P} + 81 = \pm\sqrt{10764961} = \pm 3281$

$8\text{P} = 3200$; $\qquad \text{P} = 400$.

That is, $x^2y^2 = 400$

$xy = 20$.

But, $x^2 + y^2 = \frac{820}{xy} = 41$;

and, $x^2 - y^2 = \quad 9$.

Sum, $2x^2 = 50$, and $x = 5$

$xy = 20$; $5y = 20$; $y = 4$.

88. Let $x =$ the horses; $x + 3 =$ the cows. After his purchase and sale he had $(x - 3)$ horses, and $(x + 5)$ cows.

$$\frac{x+5}{5} = x - 3$$

$$x + 5 = 5x - 15;\ 4x = 20;\ x = 5.$$

89. Let $x =$ the number of boys; then the charge to each was $\frac{1200}{x}$. If the number had been $x + 2$, the charge to each would have been $\frac{1200}{x+2}$.

Whence $$\frac{1200}{x} = \frac{1200}{x+3} + 30$$

$$\frac{40}{x} = \frac{40}{x+2} + 1$$

$$40x + 80 = 40x + x^2 + 2x;$$

whence, $x^2 + 2x + 1 = 81.$

$$x + 1 = \pm 9;\ \text{and } x = 8.$$

90. Let $x =$ the rate per hour that the man can row his boat, and $y =$ the rate of the flow of the river.

NOTE. Going down the stream, the current is with the rower; up the river it is against him.

$$2\tfrac{1}{2}x + 2\tfrac{1}{2}y = 15 \qquad (1)$$

$$7\tfrac{1}{2}x - 7\tfrac{1}{2}y = 15 \qquad (2)$$

Double (1) and (2), $$5x + 5y = 30 \qquad (3)$$

$$15x - 15y = 30 \qquad (4)$$

Divide (4) by 3, $$5x - 5y = 10 \qquad (5)$$

Add (3) and (5), $$10x = 40;\ x = 4$$

Subtract (5) from (3), $$10y = 20;\ y = 2.$$

91. The first quantity reduces to $\frac{x^2 - x}{x - 3}$. The 2d to $\frac{x^2 - 5x}{x - 3}$. Dividing as required, $\frac{x^2 - x}{x^2 - 5x}$; or, $\frac{x-1}{x-5}$. Now, write $5\frac{1}{2}$ for $x$, and we have $\frac{4\frac{1}{2}}{\frac{1}{2}} = \frac{9}{1} = 9$, *Ans.*

92. The formula for this problem is

$$(1.03)^{10} \times 200 = \$268.78 \text{ nearly.}$$

93. This problem amounts to this, that we are required to simplify and add the complex fractions

$$\frac{\dfrac{4ab}{a+b} + 2a}{\dfrac{4ab}{a+b} - 2a} + \frac{\dfrac{4ab}{a+b} + 2b}{\dfrac{4ab}{a+b} - 2b}.$$

By dividing the terms of the first complex fraction by $2a$, and of the second by $2b$, we have

$$\frac{\dfrac{2b}{a+b} + 1}{\dfrac{2b}{a+b} - 1} + \frac{\dfrac{2a}{a+b} + 1}{\dfrac{2a}{a+b} - 1}.$$

Now, multiply the terms of both fractions by $(a + b)$, and there results

$$\frac{3b+a}{b-a} + \frac{3a+b}{a-b}.$$

Changing the signs in both terms of the 1st fraction will not change its value; therefore,

$$\frac{-a-3b}{a-b} + \frac{3a+b}{a-b} = \frac{2a-2b}{a-b} = 2, \textit{ Ans.}$$

94. A geometrical mean between two quantities is the square root of the product. The product is

$$48a^5b^{\frac{5}{6}}x^{\frac{1}{4}};$$

Square root, $4\sqrt{3} \times a^{\frac{5}{2}}b^{\frac{5}{12}}x^{\frac{1}{8}} = 4a^2\sqrt{3a} \times 6^{\frac{5}{12}}x^{\frac{1}{8}}.$

95. Let $x =$ the sum; then, the sums taken were $\frac{x}{2}-20$, $\frac{x}{3}-30$, and $\frac{x}{4}40$.

Hence, $x=\frac{x}{2}+\frac{x}{3}+\frac{x}{4}-90=\frac{13x}{12}-90=1080.$

96. Let $x-3y$, $x-y$, $x+y$, and $x+3y$, represent the numbers.

$$(x-3y)(x+y)=27 \qquad (1)$$
$$(x+3y)(x-y)=72; \qquad (2)$$

or, $$x^2-3xy+xy-3y^2=27 \qquad (3)$$
$$x^2+3xy-xy-3y^2=72. \qquad (4)$$

Sum of (3) and (4), $2x^2-6y^2=99. \qquad (5)$

Difference of (3) and (4), $4xy=45. \qquad (6)$

From (5), $2x^2=99+6y^2.$

From (6), $16x^2=\frac{(45)^2}{y^2};$

whence, $8\times 99+48y^2=\frac{(45)^2}{y^2}.$

Dividing by 3, and clearing of fractions.

$$(16y^2+8\times 33)y^2= 15\times 45 = 675$$

$$16\times 16y^4+264y^2+\overline{264}^2=16\times 675+\overline{264}^2=112896.$$

Square root, $32y^2+264=\pm 336$; $4y^2=9$; $2y=3$.

But $2y$ is the common difference.

From (6), we have $(2x)(2y)=45$; or, $6x=45$

$2x=15$; $x=7\frac{1}{2}$.

Hence, the numbers must be 3, 6, 9, and 12.

97. Let $x =$ the capital.

Capital at the close of the 1st year, $\frac{115}{100}x = \frac{23}{20}x$.

This must be increased by $\frac{1}{5}$; that is, multiply by $1\frac{1}{5} = \frac{6}{5}$.

Capital at the close of the 2d year, $\frac{6}{5} \times \frac{23}{20} \times x$.

This must be increased by $\frac{1}{4}$; that is, multiply by $1\frac{1}{4} = \frac{5}{4}$.

Capital at the close of 3d year, $\frac{5}{4} \times \frac{6}{5} \times \frac{23}{20}x$.

Then, $$\frac{5}{4} \times \frac{6}{5} \times \frac{23}{20} \times x - x = 1000.5.$$

That is, $$\frac{69x}{40} - x = 1000.5$$

$$29x = 40020; \text{ and } x = \frac{40020}{29} = 1380, \textit{ Ans.}$$

98. Let $(x - y)$, $x$, and $(x + y)$, represent the numbers.

Sum, $3x = 15$; $x = 5$.

Product, $x(x^2 - y^2) = 5(25 - y^2) = 80$

$$25 - y^2 = 16;\ 9 = y^2;\ \text{and } y = 3.$$

99. Let $x - y$, $x$, and $x + y$, represent the numbers as before.

$$(x - y)^2 + x^2 + (x + y)^2 = 2900.$$

Expanding and adding, $3x^2 + 2y^2 = 2900$

$$x^2 - y^2 = x^2 - 100;\ y^2 = 100$$

$$y = 10.$$

Then, $3x^2 + 200 = 2900$; $x^2 = 900$; $x = 30$.

100. Let $x =$ the number.

Then, $\frac{x-4}{3} = 7$; $x - 4 = 21$; $x = 25$.

101. Let $x =$ the number.

Then, $x + 1 : x + 11 :: 1 : 3$

$3x + 3 = x + 11$; $2x = 8$; $x = 4$.

102. Multiply by $(a + b)^2$, and

$$\frac{a^3b}{a+b} + 2abx + b^2x = a^2x + 2abx + b^2x;$$

whence, $x = \frac{ab}{a+b}$.

103. The equation is, $40 - x = 100 - 3x$; $x = 30$.

104. Let $x^2$, and $y^2$, represent the numbers.

Then, $x^2 + y^2 = 400$, (1)

and, $x + y = 28$. (2)

Square (2), $x^2 + 2xy + y^2 = (20 + 8)^2 = 400 + 2 \times 160 + 64$.

Subtracting (1), $2xy = 320 + 64 = 384$. (3)

Subtracting (3) from (1), $x^2 - 2xy + y^2 = 16$.

$$\left.\begin{array}{ll}\text{Square root,} & x - y = \pm 4 \\ \text{But (2),} & x + y = 28\end{array}\right\} \text{whence,} \begin{array}{l} x = 16, \text{ or, } 12. \\ y = 12, \text{ or, } 16. \end{array}$$

$x^2 = 256$; and $y^2 = 144$, *Ans.*

105. The horse was bought for $x$ dollars.

The gain was $a - x$ dollars.

Then, by the condition, we have

$$x : a - x :: 100 : x$$

$$x^2 + 100x = 100a;$$

whence, $x^2 + 100x + 2500 = 100a + 2500 = 100(a + 25)$.

Square root $\qquad x. = \pm 10\sqrt{a + 25} - 50$, *Ans.*

106. Let $x =$ the first number, and $y$ the ratio.

Then $x$, $xy$, $xy^2$, are the numbers.

Product, $\qquad x^3y^3 = 1728$ (1); $x + xy^2 = 40$. (2)

Cube root of (1), $\qquad xy = 12$, the second number; (3)

whence, $\qquad 12y =$ the third number, and (2) becomes

$$x + 12y = 40.$$

But from (3), $x = \dfrac{12}{y}$; whence, $\dfrac{12}{y} + 12y = 40$

$$3 + 3y^2 = 10y; \quad y = 3.$$

107. Let $mx$ and $nx$ represent the numbers.

Then, $\qquad m^2x^2 - n^2x^2 = d^2$; $(m^2 - n^2)x^2 = d^2$.

Square root, and divide,

$$x = \frac{d}{\sqrt{m^2 - n^2}};\ mx = \frac{md}{\sqrt{m^2 - n^2}},\ \text{one number.}$$

108. Let $x$, $xy$, $xy^2$, $xy^3$, represent the numbers.

Then, $\qquad x + xy + xy^2 + xy^3 = 85$; (1)

and, $\qquad x + xy : xy^2 + xy^3 :: 1 : 16$;

or, $\qquad 1 : y^2 :: 1 : 16$; whence, $y = 4$.

Equation (1) is $\qquad x + 4x + 16x + 64x = 85$.

That is, $\qquad 85x = 85$; and $x = 1$.

109. Let $5x =$ the perpendicular, and $7x =$ the hypotenuse.

$$49x^2 = 25x^2 + 400; \; 24x^2 = 400;$$

$$4x^2 = 66\tfrac{2}{3}; \; 2x = \sqrt{66\tfrac{2}{3}}; \; 5x = \frac{5}{2}\sqrt{66\tfrac{2}{3}}.$$

110. Let $r$ represent the ratio.

Then the sum of the series $= \dfrac{3744r - 2}{r - 1}$.

The quantities $2$, $2r$, $2r^2$, $2r^3$ $2r^4$, represent five of the numbers.

Then, $\quad 2r - 2 : 2r^3 - 2r^2 \; : : 1 : 9;$

$$1 : r^2 : : 1 : 9; \text{ whence, } r = 3.$$

This value, 3, substituted in the expression for the sum,

gives $\quad \dfrac{3744 \times 3 - 2}{2} = \overline{1872 \times 3} - 1 = 5615$, *Ans.*

111. Let $x + y =$ the greater number, and $x - y =$ the less.

Their sum is $2x$, and the sum of their squares, $2x^2 + 2y^2$;

whence $\quad 2x^2 + 2y^2 + 2x = 18;$

or, $\quad x^2 + y^2 + x = 9. \; (1)$

The product is $x^2 - y^2$; whence, $x^2 - y^2 = 6. \; (2)$

Sum of (1) and (2), $\quad 2x^2 + x = 15$

Completing square and extracting square root,

$$4x + 1 = \pm 11$$

$$x = \tfrac{5}{2}, \text{ or } -3;$$

and from (2), $\quad y = \pm \tfrac{1}{2}, \text{ or } \pm \sqrt{3}.$

Hence, $\quad x + y = \tfrac{6}{2} = 3, \text{ or } -3\tfrac{1}{2};$

and, $\quad x - y = \tfrac{4}{2} = 2, \text{ or } -2\tfrac{1}{2}.$

(307, 308)

112. Let $x$ and $y$ represent the numbers.

$$x + y : x - y :: 4 : 1; \text{ or, } x + y = 4x - 4y \qquad (1)$$

$$5y = 3x,\ 125y^3 = 27x^3.$$

$$x^3 + y^3 = 152 \qquad (2)$$

Multiply (2) by 27,

and, $\quad 27x^3 + 27y^3 = 152 \times 27.$

That is, $\quad 125y^3 + 27y^3 = 152 \times 27$

$$152y^3 = 152 \times 27;\ y^3 = 27, \text{ and } y = 3.$$

113. When the 9 terms are found, the series will consist of 11 terms, of which 36 is the last term.

By the formula $\quad L = a + (n - 1)d.$

But, $\quad L = 36,\ a = 6, \text{ and } n - 3 = 10.$

Therefore, $\quad 36 = 6 + 10d, \text{ and } d = 3.$

Now, 3 added to 6 gives the next term, and so on, giving 9, 12, 15, 18, 21, etc., for the means.

115. For the solution of this problem, we have the formules,

$$L = a + (n - 1)d \qquad (1)$$

$$S = (a + L)\frac{n}{2} \qquad (2)$$

Here, $a = .034$, $d = .0004$, $S = 2.748$, and $n$ is sought.

$$L = .034 + .0004(n - 1)\ .0004 = .0336 + .0004n$$

$$a = \qquad\qquad = .034$$

$$\overline{(a + L) = \qquad\qquad .0676 + .0004n}$$

Whence, (2) becomes $(.0676 + .0004n)\frac{n}{2} = 2.748$

$$.0004n^2 + .0676n = 5.4960;$$

or, $$4n^2 + 676n = 54960$$

$$n^2 + 169n = 13740$$

$$n^2 + 169n + \left(\frac{169}{2}\right)^2 = 20880.25$$

$$n + 84.5 = \pm 144.5, \text{ and } n = 60, \textit{ Ans.}$$

116. The formula for the solution of this problem is,

$$L = a + (n - 1)d$$

$230 = a$, $13 = n$, and $13.8$ (1 year's interest) $= d$.

Hence, $\quad L = 230 + 12 \times 13.8 = 395.6.$

117. Here, $L = 3 + (n - 1)\frac{1}{3}$, and $S = \left(6 + (n - 1)\frac{1}{3}\right)\frac{n}{2}$.

$$3S = (17 + n)\frac{n}{2}, \text{ and } S = (17 + n)\frac{n}{6}, \textit{ Ans.}$$

118. Let P represent the principal.

Then the amount for 5 years is $\frac{35P}{100} + P = 317.79;$

$$135P = 31779;$$

$$P = 235.40.$$

119. Let $y$ represent the ratio.

Then, $\quad 6y^6 = 4374$; whence, $y^6 = 729$.

Square root, $\quad y^3 = 27$; $y = 3$, ratio.

120. Let P = the principal, and A the amount.

Then, $\frac{6P}{100}$ = interest for 1 year,

and, $\frac{36P}{100} + P = \frac{136P}{100} = A$;

$P = 175\frac{1}{2}$; whence,

$$\frac{136 \times 175\frac{1}{2}}{100} = \$238.68, \textit{ Ans.}$$

121. Let $x$, $xy$, $xy^2$, $xy^3$ represent the four numbers.

Then, $x + xy^3 = 35$; (1)

and, $xy + xy^2 = 30$. (2)

Dividing (1) by (2), $\frac{1+y^3}{(1+y)y} = \frac{1-y+y^2}{y} = \frac{7}{6}$. (3)

$$6 - 6y + 6y^2 = 7y;\text{ and } 6y^2 - 13y = -6$$

$$4 \times 36y^2 - (\ ) + 169 = 169 - 144 = 25$$

$$12y - 13 = \pm 5;\text{ and } y = \tfrac{3}{2}.$$

122. The formula for the solution of this problem is,

$$L = ar^{n-1}$$

$$875 = a,\ 1.06 = r,\text{ and } 7 = n.$$

Hence, $L = 875 \times 1.06^6 = 1241.20 +$.

123. Let $x$ and $y$ represent the numbers.

Then, $3xy = (x - y)(x^2 - y^2)$ (1)

$45x^2y^2 = (x^4 - y^4)(x^2 - y^2)$ (2)

(2) divided by (1), $15xy = (x^2 + y^2)(x + y)$ (3)

Multiply (1) and (3) as indicated,

and, $3xy = x^3 - x^2y - xy^2 + y^3$ (4)

$15xy = x^3 + x^2y + xy^2 + y^3$ (5)

Diff., $12xy = 2x^2y + 2xy^2$

Dividing by $2xy$, $6 = x + y$ (6)

Add (4) and (5), $18xy = 2x^3 + 2y^3$

$9xy = x^3 + y^3$ (7)

$x^2 - xy + y^2 = \frac{3xy}{2}$, by dividing (7) by (6). (8)

Square (6), $x^2 + 2xy + y^2 = 36$ (9)

Subtract (8) from (9), $3xy = 36 - \frac{3xy}{2}$; and $xy = 8$. (10)

From (6), $x = 6 - y$, and from (10) $x = \frac{8}{y}$;

whence $6 - y = \frac{8}{y}$; $y^2 - 6y = -8$; and $y = 4$, or $2$.

124. Here $L = ar^{n-1}$, as in Problem 122.

$300 = a$, $1.03 = r$, and $7 = n$.

Hence, $L = 300 \times 1.03^6 = 358.2156$.

125. Let $x =$ the breadth of the plat of land,
and, $x + 4 =$ its length.

Then, $x^2 + 4x =$ the square rods, and $4x + 8 =$ the perimeter.

Whence, $x^2 = 8$; $x = \sqrt{8}$.

126. Let $x$ = the side of the cube.

Then, $x^3$ = the solid contents.

$\sqrt{3x^2}$ = the distance from any corner to the extreme opposite corner. Whence, $x^3 = x\sqrt{3}$; $x^2 = \sqrt{3}$; $x = (3)^{\frac{1}{4}}$.

127. Let $x$ and $y$ represent the numbers.

Then, $xy = 4(x - y)$ (1)

$x^2 - y^2 = 9(x + y)$ (2)

Divide (2) by $(x + y)$, $x - y = 9$ (3)

Now, (1) becomes $xy = 36$ (4)

Square (3), $x^2 - 2xy + y^2 = 81$

But, $4xy = 144$

Sum, $x^2 + 2xy + y^2 = 225$; whence, $x + y = 15$ (5)

From (3) and (5) we find, $x = 12$, and $y = 3$.

128. Let $x$ = A's eggs, and P the price of each.

Also, let $y$ = B's eggs, and Q the price of each.

Then, $x + y = 90$ (1)

$Px = Qy$ (2)

$Py = 32$ (3)

$Qx = 50$ (4)

Divide (4) by (3), and $\frac{Q}{P}\left(\frac{x}{y}\right) = \frac{50}{32} = \frac{25}{16}$. (5)

From (2), we obtain $\frac{x}{y} = \frac{Q}{P}$. This value of $\frac{Q}{P}$, put in (5), gives

$$\left(\frac{x}{y}\right)\left(\frac{x}{y}\right) = \frac{25}{16}; \text{ or, } \frac{x^2}{y^2} = \frac{25}{16}; \text{ whence, } x = \frac{5y}{4}.$$

This value of $x$ substituted in (1), and

$$\frac{5y}{4} + y = 90$$

$$5y + 4y = 90 \times 4; \text{ whence, } y = 40, \text{ and } x = 50.$$

129. Let $x$ and $y$ represent the numbers.

$$x^2 - y^2 = 12 \qquad (1)$$

$$3x^2 - 2xy = 32 \qquad (2)$$

Let $x = vy$; then (1) becomes

$$v^2y^2 - y^2 = 12; \qquad (3)$$

and (2) becomes $3v^2y^2 - 2vy^2 = 32. \qquad (4)$

$$y^2 = \frac{12}{v^2 - 1}, \text{ and } y^2 = \frac{32}{3v^2 - 2v}; \qquad (5)$$

whence, $\frac{3}{v^2 - 1} = \frac{8}{3v^2 - 2v}$

$$9v^2 - 6v = 8v^2 - 8$$

$$v^2 - 6v = -8; \; v^2 - (\;) + 9 = 1$$

$$v - 3 = \pm 1; \text{ and } v = 4, \text{ or } 2.$$

Substituting these values of $v$ in (5), and

$$y^2 = \frac{12}{15}, \text{ or } \frac{12}{3}; \; y^2 = \frac{4}{3}, \text{ or } 4. \text{ Whence, } y = 2, \text{ and } x = 4.$$

**180.** As the number of terms is odd, let $x =$ the middle term, and let $y =$ the common difference.

The series is $x - 2y$, $x - y$, $x$, $x + y$, and $x + 2y$.

Sum, $5x = 25$; whence, $x = 5$

The product of $x - y$ and $x + y$ is $x^2 - y^2$.

The product of $x - 2y$ and $x + 2y$ is $x^2 - 4y^2$.

Whence, $$5(x^2 - y^2)(x^2 - 4y^2) = 945$$

$$(x^2 - y^2)(x^2 - 4y^2) = 189$$

$$x^4 - x^2y^2 - 4x^2y^2 + 4y^4 = 189.$$

But, $$x^4 = 625;$$

whence, $$625 - 125y^2 + 4y^4 = 189;$$

$$4y^4 - 125y^2 = -436$$

Complete square and extract square root, and

$$8y^2 - 125 = \pm 93;\ 8y^2 = 32;\ \text{and } y = 2.$$

**181.** Let $x =$ the greater measure, and $y =$ the less.

$$xy = 40 \qquad (1)$$

$$(2x + y)(x + 2y) = 432 \qquad (2)$$

$$2x^2 + 5xy + 2y^2 = 432$$

Subtract (1), $2x^2 + 4xy + 2y^2 = 392$

Divide by (2) and extract square root, and

$$x + y = 14 \qquad (3)$$

From (1), $x = \frac{40}{y}$, and from (3), $x = 14 - y$;

Hence, $\frac{40}{y} = 14 - y$; $y^2 - 14y = -40$; and $y = 10$, or 4.

132. Let $x =$ the working days, and $y =$ the idle days.

Then, $$150x - 65y = 0$$

$$x + y = 129$$

$$65x + 65y = 129 \times 65$$

$$215x = 129 \times 65 = 8385.$$

Whence, $$x = 39, \text{ and } y = 90.$$

133. Let $x$, $y$, and P represent the numbers.

$$xy = 6 \quad (1)$$

$$xP = 8 \quad (2)$$

$$yP = 12 \quad (3)$$

Product of (1) and (2), $x^2Py = 48.$

That is, $$12x^2 = 48;$$

whence, $$x^2 = 4; \text{ and } x = 2.$$

134. Let P = the perpendicular. In every triangle half the base multiplied into the perpendicular, is equal to the area.

Therefore, $25P = 600 = 6 \times 100;$

and, $P = 6 \times 4 = 24$, the perpendicular.

Let S represent the shorter side adjacent to this perpendicular.

Then $S + 10 =$ the longer side.

Now, by the property of right angled triangles,

$\sqrt{S^2 - 24^2}$ = one segment of the base;

$\sqrt{(S + 10)^2 - 24^2}$ = the other segment; whence,

$$\sqrt{(S + 10)^2 - 24^2} + \sqrt{S^2 - 24^2} = 50;$$

$$\sqrt{S^2 + 20S + 100 - 576} = 50 - \sqrt{S^2 - 576}.$$

By squaring,

$$S^2 + 20S - 476 = 2500 - 100\sqrt{S^2 - 576} + S^2 - 576.$$

Reducing, $20S = 2400 - 100\sqrt{S^2 - 576}$.

Transposing, &c., $5\sqrt{S^2 - 576} = 120 - S$.

Squaring, $25S^2 - 25 \times 576 = 14400 - 240S + S^2$;

$$24S^2 - 25 \times 576 = 14400 - 240S.$$

Dividing by 24, $S^2 - 25 \times 24 = 600 - 10S$

$$S^2 + 10S = 1200$$

$$S^2 + (\ \ ) + 25 = 1225$$

$$S + 5 = \pm 35$$

$$S = 30; \text{ or, } -40.$$

The side cannot be $-40$; therefore $S = 30$, and $S + 10 = 40$.

135. Let $x$, $y$, $t$, $n$, represent 4 numbers in the order here written.

Then,

$$xyt = a \qquad (1)$$

$$xyn = b \qquad (2)$$

$$xtn = c \qquad (3)$$

$$ytn = d \qquad (4)$$

Product, $x^3y^3t^3n^3 = abcd$;

$$xytn = (abcd)^{\frac{1}{3}}$$

But (1), $\quad xyt = a.$

By division, $\quad n = \dfrac{\sqrt[3]{abcd}}{a}.$

By a similar process we can find $t$, $y$, and $x$ in succession.

186. Let $t$ = B's time, and $11 + 4t$ = B's distance.

B travels during the last hour, $4\frac{1}{2} + (t-1)\frac{1}{4}$.

And B's whole distance is represented by $(9 + (t-1)\frac{1}{4})\,\frac{t}{2}$

Therefore, $\left(9 + \frac{t}{4} - \frac{1}{4}\right)\frac{t}{2} = 11 + 4t$

$$(36 + t - 1)\,\frac{t}{2} = 44 + 16t$$

$$t^2 + 35t = 88 + 32t$$

$$t^2 + 3t = 88;\ \text{whence, } t = 8.$$

187. Let 6 = the least side; $x$ = the common difference; $6 + x$ = the perpendicular;
and $6 + 2x$ = the hypotenuse.

Then, $\quad 36 + (6 + x)^2 = (6 + 2x)^2$

$$36 + 36 + 12x + x^2 = 36 + 24x + 4x^2$$

$$36 = 12x + 3x^2;\ \text{or, } x^2 + 4x = 12$$

$$x^2 + (\ ) + 4 = 16$$

$$x + 2 = 4;\ \text{and } x = 2;$$

whence, 6, 8, and 10, are the sides.

If the perpendicular is the least side and $x =$ the common difference, then $6 - x =$ the perpendicular, and $6 + x =$ the hypotenuse.

$$\text{Whence,} \quad (6 + x)^2 = (6 - x)^2 + 36$$

$$36 + 12x + x^2 = 36 - 12x + x^2 + 36$$

$$24x = 36;\ 2x = 3;\ \text{and } x = 1\tfrac{1}{2};$$

whence, $4\frac{1}{2}$, 6, and $7\frac{1}{2}$, are the sides.

188. Let $x^2$ and $y^2$ represent the two squares.

$$x^2 + y^2 = 52 \qquad (1)$$

Divide (2) by $2(x + y)$,

$$2x^2 - 2y^2 = 4(x + y); \qquad (2)$$

$$x - y = 2.$$

$$\text{By squaring,} \quad x^2 - 2xy + y^2 = 4. \qquad (3)$$

$$\text{Subtracting (3) from (1), and} \quad 2xy = 48. \qquad (4)$$

$$\text{Add (4) and (1),} \quad x^2 + 2xy + y^2 = 100.$$

$$x + y = 10$$

But, $x - y = 2$; therefore $x = 6$, $x^2 = 36$, and $y^2 = 16$.

189. Let $x =$ the tens, and $y =$ the units in the number.

$$\text{Then,} \quad 10x + y = \text{the number.}$$

$$10x + y = 2xy \qquad (1)$$

$$10x + y + 27 = 10y + x \qquad (2)$$

$$\text{(2) reduced,} \quad 9x + 27 = 9y$$

$$x + 3 = y.$$

This value of $y$ substituted in (1), gives

$$11x + 3 = 2x(x + 3)$$

$$2x^2 - 5x = 3;\ \text{whence, } x = 3;\ y = 6.$$

**140.** Let $x - y$, $x$, and $x + y$, represent the numbers.

$$3x = \frac{3}{2}; \text{ and } x = \frac{1}{2}, \text{ the 2d number.}$$

$$\frac{1}{\frac{1}{2} - y} + 2 + \frac{1}{\frac{1}{2} + y} = 7\tfrac{1}{3}$$

$$\frac{2}{1 - 2y} + \frac{2}{1 + 2y} = \frac{16}{3}; \text{ or, } \frac{1}{1 - 2y} + \frac{1}{1 + 2y} = \frac{8}{3};$$

$$1 + 2y + 1 - 2y = \frac{8}{3}(1 - 4y^2);$$

or,
$$1 = \frac{4}{3}(1 - 4y^2);$$

$$3 = 4 - 16y^2; \; 16y^2 = 1; \; 4y = 1; \; y = \tfrac{1}{4};$$

whence, $\frac{1}{2} - \frac{1}{4} = \frac{1}{4} =$ 1st number, and $\frac{3}{4} =$ the 3d

141. When the three means are found, the series will consist of five terms; hence, for the last term we must raise the ratio to the 4th power, and multiply it by the first term.

Let $r$ be the ratio; then,

$$\frac{1}{2}r^4 = \frac{2}{9}; \text{ or, } r^4 = \frac{4}{9};$$

$$r^2 = \tfrac{2}{3}; \; r = \sqrt{\tfrac{2}{3}} = \tfrac{1}{3}\sqrt{6}.$$

Hence, the second term must be $\frac{1}{2} \times \frac{1}{3}\sqrt{6} = \frac{1}{6}\sqrt{6}$.

142. Let $x$, $y$, and $z$ be the three quantities.

Then,
$$x^2 + y^2 + x + y = 18 \quad (1)$$
$$x^2 + z^2 + x + z = 26 \quad (2)$$
$$y^2 + z^2 + y + z = 32 \quad (3)$$

From the sum of (1) and (2) subtract (3), and we shall obtain

$$2x^2 + 2x = 12$$

$$x^2 + x = 6$$

$$4x^2 + (\;) + 1 = 25;\ 2x + 1 = \pm 5;$$

whence, $x = 2$, or $-3$.

These values of $x$ substituted in (1) and (2), enable us to find $y$ and $z$.

143. Let P be any principal, and $r$ the rate of interest decimally expressed. Then the interest for the 1st year is $Pr$; add this to P for the principal the 2d year.

Then, $P + Pr$, or $P(1 + r)$ = principal the 2d year.

Multiply by $r$

Interest the 2d year, $Pr(1 + r)$.

Principal and interest added, $P(1 + r) + Pr(1 + r)$;

or, $(P + Pr)(1 + r)$;

or, $P(1 + r)^2$ = the amount for 2 years.

Then, $P(1 + r)^3$ = " " 3 "

$P(1 + r)^4$ = " " 4 "

&c. = &c.

To apply this to the present problem, we observe that the first year's interest is $Pr$.

The interest for the third year is

Amount — Principal.

Hence, $Pr : P(1 + r)^3 - P(1 + r)^2 : : 25 : 36$;

or, $r : (1 + r)^3 - (1 + r)^2 : : 25 : 36$;

$r : r(1 + r)^2 : 25 : 36$

$1 : (1 + r)^2 : : 25 : 36.$

Square root, $1 : 1 + r :: 5 : 6$;

whence, $5r = 1$; or, $r = 0.2$; or, 20 per cent.

$$P(1 + r)^3 - P = 364$$

$$P(\overline{1 + r}^3 - 1) = 364$$

$$P(3r + 3r^2 + r^3) = 364$$

But,
$$\begin{aligned} r &= .2 \\ \hline 3r &= .6 \\ 3r^2 &= .12 \\ r^3 &= .008 \\ \hline \end{aligned}$$

Sum, $= .728$; that is, $.728P = 364$;

whence, $P = 500$.

144. Let $x - y$, $x$, and $x + y$, represent the numbers.

$$3x = 36, \text{ and } x = 12, \text{ the sum.}$$

$$x^2 - y^2 + 4 = x^2; \text{ whence, } y = 2.$$

145. Let $x$, $xy$, and $xy^2$ = the three numbers.

$$x + xy + xy^2 = 52 \qquad (1)$$

$$x + xy^2 : x^2y^2 :: 10 : 36 \qquad (2)$$

$$1 + y^2 : xy^2 :: 10 : 36$$

$$x = \frac{(1 + y^2)36}{10y^2}.$$

From (1), $x = \dfrac{52}{1 + y + y^2} = \dfrac{52}{(1 + y^2) + y}$;

whence, $\dfrac{26}{(1 + y^2) + y} = \dfrac{(1 + y^2)18}{10y^2}$

$$260y^2 = 18(1 + y^2)^2 + 18y(1 + y^2).$$

This is an equation of a higher degree which the pupil has not yet learned to solve, therefore we must take some other solution.

Let $x^2$, $xy$, and $y^2$, represent the numbers.

Then, $$x^2 + xy + y^2 = 52; \qquad (1)$$

and, $$x^2 + y^2 : x^2y^2 :: 10 : 36. \qquad (2)$$

From (1) we obtain $x^2 + y^2 = 52 - xy$, and this value of $x^2 + y^2$ substituted in (2), produces,

$$52 - xy : x^2y^2 :: 10 : 36;$$

whence, $$10x^2y^2 = 52 \times 36 - 36xy$$

$$5x^2y^2 + 18xy = 26 \times 36$$

$$20 \times 5x^2y^2 + (\ ) + \overline{18}^2 = 20 \times 26 \times 36 + \overline{18}^2 = 19044;$$

$$10xy + 18 = 138; \text{ and } xy = 12, \text{ the middle term.}$$

Now, from (1) we obtain

$$\left.\begin{aligned} x^2 + y^2 &= 40; \\ \text{also,}\quad 2xy &= 24. \end{aligned}\right\} \text{Adding and subtracting, and equating square roots, and}$$

$$\left.\begin{aligned} x + y &= \pm 8 \\ x - y &= \pm 4 \end{aligned}\right\} \text{whence,} \left\{\begin{aligned} x &= 6, \text{ or } 2;\ x^2 = 4, \text{ or } 36. \\ y &= 2, \text{ or } 6;\ y^2 = 36, \text{ or } 4. \end{aligned}\right.$$

146. Let $x$, $xy$, and $xy^2$, represent the numbers.

Then $x^3y^3 = 1$; whence, $xy = 1$, the middle number.

Whence, by the 2d condition,

$$x - 1 : 1 - xy^2 :: 1 : 3$$

$$3x - 3 = 1 - xy^2.$$

But, $$xy^2 = y; \text{ because } xy = 1.$$

That is, $$3x - 3 = 1 - y$$

$$3x + y = 4.$$

Multiply by $x$, and $3x^2 + xy = 4x$;

that is, $3x^2 - 4x = -1$

$$36x^2 - (\ \ ) + 16 = 16 - 12 = 4$$

$$6x - 4 = \pm 2;\quad x = 1, \text{ or } \tfrac{1}{3}.$$

147. Let $x$, $xy$, $xy^2$, represent the numbers.

Then $3x$, $2xy$, and $xy^2$, are in arithmetical progression.

Whence, $3x + xy^2 = 4xy$.

Divide by $x$, $3 + y^2 = 4y$; whence, $y = 3$.

Again, $x$ .. $xy + 8$ .. $xy^2$, form an arithmetical progression.

Whence, $x + xy^2 = 2xy + 16$.

But $y = 3$; therefore, $10x = 6x + 16$, $4x = 16$, $x = 4$, the first term.